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Values of Non-Atomic Games, Part V: Monetary Economies

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PREFACE

This Report concludes a five-part series on the mathematical theory of non-atomic games. (See RM-5468-PR, RM-5842-PR, RM-6216, and RM-6260.) The term "non-atomic," borrowed from measure theory and probability theory, signifies that in these games with infinitely many participants, no single individual is big enough to influence the outcome by himself. Such games have served as mathematical models for large-scale competitive systems in economics or politics. In this Report the applications of the theory to a class of economics models are developed.

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SUMMARY

The value of a multiperson game is a function that associates to each player a number that, intuitively speaking, represents an a priori evaluation of what it is worth to play the game from his position. A non-atomic game is a special kind of infinite-person game in which no individual player has significance. The value concept was originally defined only for finite-person games; in Parts I-III of this series several approaches to the problem of extending the value concept to non-atomic games were developed. In Part IV the relationship of the value to another solution concept--the core--was considered.

In the present Part V, the results of the previous parts are applied to a certain class of basic economic models, interpretable either as exchange economies with money or as productive economies. The general conclusion, which takes a number of specific forms, is that under fairly wide conditions the value of the game derived from such a model exists and coincides with the unique payoff distribution in the core of the game, as well as with the unique payoff distribution associated with the competitive equilibrium or equilibria of the underlying model. This exact agreement of several solutions, in an infinite-person setting, may be compared with known results on the convergence of these solutions in the limit, in similar models with large but finite numbers of participants.

CONTENTS

28. INTRODUCTION TO PART V

This is the fifth in a series of papers with the overall title "Values of Non-Atomic Games".* Familiarity with the previous parts will be assumed throughout. Numeration of the sections will be continued here, to enable easy reference to the previous parts. Other conventions established previously will also be maintained here.

In this part we will apply the theory developed in the previous parts to certain economic models. These models may be interpreted either as monetary exchange economies**, or as productive economies similar to—but more general than—the one described in the introduction to Part IV.*** Our chief result is that under fairly wide conditions, the game derived from such a model is in pNA, that there is a unique point in its core, and that this

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*For the previous parts, see [I, II, III, IV] in the list of references.

**I.e. "markets with money" or "markets with side payments"; cf. [S-S_1, S-S_2, S_8]. These are special cases of the more classical Walrasian exchange economies (cf., e.g., [N, D-Sca, A_1]), which may be called "markets without side payments", and which we hope to study from the value viewpoint in a subsequent paper.

***See Formula (26.1) and the subsequent discussion. Incidentally, the word "monetary" in the title of this part refers only to the first interpretation; the production interpretation is not connected with money.

unique point coincides with the value. We shall also define the notion of competitive equilibrium for such economies, and show that this too then yields a unique payoff, which also coincides with the value, and therefore with the unique core point.

Section 29 is devoted to a careful conceptual discussion of several aspects of economic models with a continuum of economic agents. This is needed for a proper understanding of Section 30, in which we introduce and motivate the particular economic model that is the subject of this paper. Section 31 contains the statement of the results concerning the relation between the core and the value. In Section 32 we will introduce and discuss the competitive equilibrium, and relate it to the previously described concepts. Section 33 is devoted to some examples, and Section 34 to a brief discussion of related literature. Sections 35 through 41 are devoted to the proofs. In the last section, Section 42, we discuss some possibilities for extensions of our results.

It is to be stressed that the proof of the main result--i.e., the membership of our game in pNA, the existence of a unique point in the core and its coincidence with the value--does not make any use of the notion of competitive equilibrium; rather, it is based directly on Theorem F in Part IV.

29. CONCEPTUAL PRELIMINARIES

In this section we would like to clarify some of the ideas used in connection with economic models with a continuum of economic agents. Specifically, we shall discuss the use of integration in connection with such models, and the ideas of payoff vector, allocation, and side-payment game in such a context. This section does not contain a discussion of the larger issues involved in the use of continuous game and economic models; for such a discussion, see [M-S] and [A_1].

Properly to understand the use of integration in connection with continuous economic models, it is convenient to use an analogy with physics, where continuous models are plentiful and well-understood. Let us recall the problem of computing (or for that matter, defining) the gravitational force exerted by a solid beam I on a given mass point x in space, whose mass is, say, M. One divides I into "small" pieces, calling a typical piece "Δs". Then if ρ is the distance function and s is a point in Δs, all points in Δs have a distance approximately $\rho(x, s)$ from x. Therefore if $\mu(\Delta s)$ denotes the mass of Δs, the gravitational force exerted by Δs on x is approximately

$$\frac{M\mu(\Delta s)(s - x)}{\rho^3(s, x)}$$

(whose magnitude is $M\mu(\Delta s)/\rho^2(s, x)$); and the total gravitational force exerted by I on x is approximately

$$\Sigma[M(s - x)/\rho^3(s, x)]\mu(\Delta s), \tag{29.1}$$

the sum being taken over all the "small" pieces into which we have divided I. When we say that Δs is "small", what we mean is that its diameter is small; precisely, what is required is that $(s - x)/\rho^3(s, x)$ be almost constant as s ranges over Δs.

The next step is to pass to the limit. As the diameters of the Δs tend to 0, the expression (29.1) tends to

$$\int_I [M(s - x)/\rho^3(s, x)]\mu(ds); \tag{29.2}$$

at the same time the approximations become better and better, and the errors involved tend to 0. Hence we conclude that the total force exerted by I on x is in fact precisely the integral (29.2).

There is also a slightly different way of looking at the integral (29.2). One thinks of I as being divided into

"infinitesimal pieces" ds, each with an "infinitesimal mass" $\mu(ds)$. The piece ds has an infinitesimal diameter; if one wishes one can think of it as consisting of a single point, located at s. The force exerted by it on x is

$$[M(s - x)/\rho^3(s, x)]\mu(ds),$$

and the total force is the "sum" of these infinitesimal forces, namely the integral (29.2).

Some readers may be disturbed by the use of terms such as "infinitesimal", which we have not properly defined.* Such readers may take the discussion in terms of infinitesimals to be simply an abbreviation for the somewhat more lengthy discussion involving "small pieces" and a limiting process. Imprecise as it may be, though, the discussion in terms of infinitesimals has a certain direct conceptual appeal, which is lacking in the limit discussion. Each infinitesimal piece ds exerts a force which can be calculated exactly—<u>not</u> approximately—by a single straightforward application of Newton's formula for the gravitational

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*This is not to say that they cannot be properly defined; cf. [Rob].

attraction between two mass points. And the total force is simply the sum of these individual forces. By comparison, the limit approach seems conceptually devious.

In the case of economic models, the "infinitesimal" approach has an additional intuitive advantage. People still think even of very large economies as consisting of individual agents; intuitively, then, such an agent can be associated with an "infinitesimal piece". In the physical analogy, one could think of our beam I as being made up of many individual mass points—as indeed it is, if one considers an atom a point. One replaces this set of mass points by a continuum—both for mathematical convenience and for a better physical understanding of the gravitational field around a beam. But in intuitive discussion of the integral, it may still be convenient to associate an "infinitesimal piece" with one of the individual mass points. It should be stressed, though, that such an association is not necessary, neither in the economic nor in the physical situation. In both situations, the infinitesimal piece can be though of as a set of individuals which has an infinitesimal mass or measure, and all of whose members have the same physical or economic properties (for example the same distance from x in the physical case, the same utility in the economic case).

In intuitive discussion in the sequel, we shall adopt the convention of associating an infinitesimal with a single individual. This is chiefly because it is easier, e.g., to write "the trader ds" rather than "the set ds of traders" or "one of the traders in ds"; if the reader wishes, he can substitute the alternative interpretation. The reader should be careful to note that we are associating an individual with an infinitesimal subset ds of I, not with a point* s in I. We will adopt the convention that the point named s is always a member of the set named ds. It will be understood that all functions of s that appear in the analysis are constant on every ds. For example, we shall describe the initial bundle of a trader ds by an expression of the form $\underset{\sim}{a}(s)\mu(ds)$; intuitively, it is to be understood that s is a point in the infinitesimal set ds, and that $\underset{\sim}{a}$ is a function on I that is constant on ds, so that it does not matter which point s in ds is chosen.

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*It may seem we are backtracking a little from the interpretation given in Section 2, where we said simply that "the members of I are players". Also in [A_1], the individual points in the continuum were called "traders"; and even in the introduction to Part IV, we referred to a "producer s". There is, however, no real change in outlook; here we are simply being more careful as regards interpretation. In the sequel it may again become convenient to refer to a point in I as a "player" or "trader", and then we shall not hesitate to do so, in spite of the loss of strict accuracy.

Readers who prefer to think of the integral in terms of the limiting process may make the necessary re-interpretations, in which ds is replaced by Δs, s is a point in Δs, and Δs is chosen so that $\underset{\sim}{a}$ is "almost constant" on it.

In closing the discussion of this physical analogy, we would like to stress that the whole discussion is concerned exclusively with the passage from the given physical situation to the mathematical model. Once one accepts the integral as properly representing the desired force, the rest of the treatment can be perfectly precise, in the best traditions of modern mathematical analysis. The situation in economics is similar; the mathematical model, once constructed, can be analyzed with the ordinary mathematical tools, with the precision that is characteristic of mathematical analysis. Only in constructing the model, and in relating it to the economic ideas that motivate its construction, is it convenient to make use of words such as "infinitesimal", and of the corresponding ideas.

Next, we would like to discuss the idea of "payoff vector" and related ideas. In a game with a finite set N of players, a payoff vector is simply a member x of E^N, i.e., a function from N to the reals; intuitively, it is

to be thought of as an outcome, where the i-th component $x(i)$ signifies the payoff* to player i.

When we are thinking in terms of coalitions rather than individuals, it is convenient to think of the payoff vector x as a measure on N, defined for all $S \subset N$ by

$$x(S) = \sum_{i \in S} x(i);$$

here $x(S)$ signifies the total payoff to S under the outcome x. This point of view is especially useful in connection with games with a continuum of players, such as we are studying in this series of papers, say games with a player space $(I, \mathcal{C})$. In such games a payoff vector may often be represented by a non-atomic measure; this means that the individual player gets only an infinitesimal payoff, whereas the total payoff to a coalition is often a

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*Depending on the context, this payoff could be in money; in a consumer product, such as the "finished good" of the production model mentioned in Part IV and further to be developed in Section 30 below; or in the "transferable utility" which may be familiar to some of our readers from n-person game theory. Regardless of the direct interpretation of the payoff, however, when we apply the notion of the value of the game (unlike some other solution concepts) we are in effect assuming that the payoff is a utility indicator of the "cardinal" kind, in the sense that the resulting solution will generally be invariant only under linear order-preserving transformations of the payoffs.

positive number. For the sake of generality,* we define a payoff vector to a game with player space $(I, \mathcal{C})$ to be any member of FA.

Having interpreted the notion of "payoff vector" in the continuous case, we now come to the notion of "game" itself. This was interpreted in Sec. 2 as a real-valued set function v. The number $v(S)$, for $S \in \mathcal{C}$, was interpreted as the "total payoff that the coalition S, if it forms, can obtain for its members", and was called the "worth" of S.

Now there are several assumptions about the nature of a game that are implicit in the use of a real-valued set function to describe it; we would like to discuss just one of them here, namely the assumption of "unrestricted side payments". This means that not only can each coalition S obtain for its members a total of $v(S)$, but that it can also distribute this total among its members in any way it pleases. Thus, if ν is any member of FA with $\nu(S) = v(S)$, then S can act so that each $T \subset S$ will obtain $\nu(T)$, or in

*We have not found it necessary to allow more generality, e.g. to allow unbounded measures. Neither is it conveneient, on the other hand, to restrict the generality, e.g. to consider only completely additive measures. This is because FA is a subspace of BV, and if we look at a member μ of FA as a game, then the payoff vectors naturally associated with this game cannot be expected to be completely additive if μ is not; for example, the core of μ consists of the unique point μ itself.

other words, so that each "member" ds of S will obtain $v(ds)$.

Throughout this paper, we shall deal only with "games with unrestricted side payments", i.e., games obeying this condition. Indeed, if this condition were not satisfied, i.e., if only certain distributions totaling $v(S)$ could be obtained by S, then the situation would not be adequately described by the function v. To describe the situation in this case, one must at least specify, for each S, exactly which distributions of payoff the coalition S can obtain for its members (cf. [A-Pel]). Such an extension of the underlying model we hope to treat in another paper.

30. DESCRIPTION OF THE MODEL AND ECONOMIC INTERPRETATION

Let Ω denote the nonnegative orthant of a Euclidean space E^n, whose dimension n will be fixed throughout. Superscripts will be used to denote coordinates. For x and y in E^n we write $x \geqq y$ if $x^i \geqq y^i$ for all i, $x \geq y$ if $x \geqq y$ but not $x = y$, and $x > y$ if $x^i > y^i$ for all i. A real-valued function f on Ω will be called <u>nondecreasing</u> if $x \geqq y$ implies $f(x) \geqq f(y)$, and <u>increasing</u> if $x \geq y$ implies $f(x) > f(y)$. The scalar product $\Sigma_{i=1}^n x^i y^i$ of two members x and y of E^n will be denoted $x \cdot y$. The symbol 0 will denote both the number zero and the origin of a Euclidean space; no confusion will result.

Let $\mu \in NA^+$; μ will be fixed throughout. For convenience we shall assume that $\mu(I) = 1$, although most of our results, and in particular all those stated in Section 31, are true without this assumption* as well. If g is a μ-integrable function on I and $S \in \mathcal{C}$, we will use the notations $\int_S g$, $\int_S g d\mu$, $\int_S g(s) d\mu(s)$, and $\int_S g(s)\mu(ds)$ interchangeably. All occur in the literature, and for different purposes one or the other will be more convenient. When the range of integration in an integral is not specified,

*The case of general $\mu(I) \neq 0$ follows trivially from that in which $\mu(I) = 1$.

it will be understood to be I; thus $\int g$ and $\int_I g(s)\mu(ds)$ are the same thing. The phrases "integrable", "almost all", and so on, will be used to mean "μ-integrable", "μ-almost all", and so on.

For each $s \in I$, let $\underset{\sim}{a}(s)$ be in Ω, and let $u(\cdot, s)$ be an increasing nonnegative real-valued function on Ω. We will be concerned with the set function v defined by

$$(30.1) \quad v(S) = \max \{\int_S u(\underset{\sim}{x}(s), s)d\mu(s) : \underset{\sim}{x}(s) \in \Omega \text{ for all } s \text{ and } \int_S \underset{\sim}{x}d\mu = \int_S \underset{\sim}{a}d\mu\},$$

the maximum being taken over all μ-integrable functions $\underset{\sim}{x}$ that satisfy the constraints. Note that the equation in the constraints is a vector equation; thus when we say that $\underset{\sim}{x}$ is μ-integrable, we mean that all its coordinates are μ-integrable. Naturally, in order that the integrals inside the curly brackets be meaningful, it is necessary to impose certain measurability and integrability conditions on the functions u and $\underset{\sim}{a}$. Furthermore, even if the integrals involved exist, it is by no means clear or even always true that the expression being maximized is bounded;

and even if it is bounded, its supremum may not be attained.*
These matters will be treated in the next section, where sufficient conditions will be imposed on u and $\underset{\sim}{a}$ to ensure that the integrals involved exist, and that the maximum exists. In this section we would like to concentrate on the economic interpretations of the set function v.

The reader will recall from the introduction that there are two economic interpretations, one in terms of monetary exchange economies and one in terms of productive economies. We would like to present the interpretation in terms of productive economies first, since it is simpler. There are n kinds of raw material, and only one kind of finished good. The space I consists of infinitesimal producers ds. Given a bundle, (i.e. vector) x in Ω of raw materials, producer ds can produce an amount $u(x, s)\mu(ds)$ of the finished good. Next, $\underset{\sim}{a}(s)\mu(ds)$ is the bundle of

*It is quite possible for the sup to exist without the max existing. We have not treated such situations. One reason is that they are conceptually somewhat slippery. It is of course possible to define v(S) by means of the sup, but the idea of the "worth" of a coalition then loses some of its intuitive force. The way we are used to thinking about core and value would presumably also need some revision. If, for example, $\nu(I) = v(I)$, and the sup in the definition of v(I) is not attained, then we cannot really think of ν as a distribution of the amount available to I, since v(I) is not really available to I. A more important reason for insisting that the sup be attained is that the mathematics would otherwise be even more complicated than it now is. For a discussion of where one is led if one replaces "max" by "sup", see Subsection D of Section 33 and Subsection D of Section 42.

raw materials initially available to the producer ds; hence the total bundle of raw materials initially available to a coalition S is $\int_S \underset{\sim}{a}(s)\mu(ds) = \int_S \underset{\sim}{a} d\mu$. Now S may reallocate this amount among its members in any way it pleases; that is, if the members of S agree, they may assign to each member ds of S an amount $\underset{\sim}{x}(s)\mu(ds)$ rather than $\underset{\sim}{a}(s)\mu(ds)$, on condition that $\underset{\sim}{x}(s) \in \Omega$ and $\int \underset{\sim}{x} d\mu = \int \underset{\sim}{a} d\mu$. Then if the maximum in (30.1) exists, and if S pools and redistributes its resources and then pools the finished goods produced by all the members, then the total amount in the resulting pool of finished goods can be as high as v(S). In short, the coalition S, if it forms, can obtain for its members a total payoff of v(S); in this sense, v(S) is the worth of S.*

In the interpretation in terms of monetary exchange economies there are n + 1 consumer goods, indexed by 0, 1, ..., n. The good indexed by 0 is called money and, unlike the others, may appear in negative as well as positive amounts. The space I consists of infinitesimal traders ds, and the amount of any good typically available to ds will also be infinitesimal; a typical bundle will have the form $(x^0, \underset{\sim}{x})\mu(ds)$, where $x^0 \in E^1$ and $x \in \Omega$. Each trader ds has a preference

*Strictly speaking, we do not have "unrestricted side payments" (see Section 29) in this interpretation, since the individual holdings of the finished good must be nonnegative. However, since v is monotonic, negative payoffs cannot occur in the value, nor, for that matter, in the core.

order on the set of all such bundles. Money enters into these preferences in a very special way; specifically, $x^0 + u(x, s)$ is a utility function for the trader ds. In other words, if (x^0, x) and (y^0, y) are in $E^1 \times \Omega$, then ds prefers $(x^0, x)\mu(ds)$ to $(y^0, y)\mu(ds)$ if and only if

$$x^0 + u(x, s) > y^0 + u(y, s).$$

The consumer ds starts out with no money and with the bundle $\underset{\sim}{a}(s)\mu(ds)$ of goods 1, ..., n. By trading, it may be possible for him to improve his position, i.e., to obtain commodity bundles which he prefers to his initial bundle. Let S be a coalition (i.e., $S \in \mathcal{C}$), and let $\underset{\sim}{x}$ be such that $\underset{\sim}{x}(s) \in \Omega$ for all s and $\int_S \underset{\sim}{x} d\mu = \int_S \underset{\sim}{a} d\mu$. This means that the members of S can trade among each other—redistribute their initial resources—in such a way that after the trade, ds will be holding the bundle $\underset{\sim}{x}(s)d\mu(s)$ of goods 1, ..., n, but still no money. The utility to consumer ds of his new bundle will be $u(\underset{\sim}{x}(s), s)\mu(ds)$, and so if we "add" the utilities of all consumers in S we will get a total of $\int_S u(\underset{\sim}{x}(s), s)\mu(ds)$. Let us choose $\underset{\sim}{x}$ so that the maximum in (30.1) is attained; then this total is exactly v(S).

Generally, adding up utilities of different consumers is an economically meaningless procedure. In this case, however, the availability of money lends significance to the total utility of S. Indeed, we claim that <u>any</u> distribution of utilities to the consumers in S whose total is v(S) is achievable by the coalition S. This means that if ν is any measure with $\nu(S) = v(S)$, then the coalition S can distribute its total bundle $(0, \int_S \underset{\sim}{a} d\mu)$ so that the utility of consumer s in S will be $\nu(ds)$. To see this, define a measure ξ by

$$\xi(U) = \nu(U) - \int_U u(\underset{\sim}{x}(s), s)\mu(ds)$$

for all $U \in \mathcal{C}$. Then $\xi(S) = 0$, i.e., ξ restricted to S is a feasible redistribution among the traders of S of the initial total—namely 0—of money available to this coalition. If S redistributes its money in this way, then each trader ds will get the bundle $(\xi(ds), \underset{\sim}{x}(s)\mu(ds))$, whose utility is

$$\xi(ds) + u(\underset{\sim}{x}(s), s)\mu(ds) = \nu(ds).$$

Thus this game satisfies the condition of "unrestricted side payments," and the worth of each coalition S is adequately described by the number v(S).

The reader is referred to [S-S_1] for a discussion of the significance of this kind of monetary exchange economy, and its relation to the more commonly employed Walrasian barter model. (See also Section 32.)

31. STATEMENT OF MAIN RESULTS

Throughout, the measure μ and the functions u and $\underset{\sim}{a}$ will be as specified at the beginning of Sec. 30, and the set-function v as defined in (30.1).

We shall say that $u(x, s) = o(\|x\|)$ as $\|x\| \to \infty$, integrably in s, if for each $\epsilon > 0$ there is an integrable function $\underset{\sim}{\eta}$ on I, such that $|u(x, s)| \leq \epsilon\|x\|$ whenever $\|x\| \geq \underset{\sim}{\eta}(s)$. If $\underset{\sim}{\eta}$ is bounded, then this is equivalent to saying that $u(x, s) = o(\|x\|)$ as $\|x\| \to \infty$, uniformly in s. But in general, the two concepts are not equivalent; for example, if $n = 1$, then $x^{\frac{1}{2}}/s^{\frac{1}{4}} = o(x)$ integrably, but not uniformly. The concept of integrable convergence was introduced in [A–P] in order to deal with the question of the existence of the maximum in expressions of the form (30.1).

The function u will be called Borel-measurable if it is measurable on the product σ-field $\mathcal{B} \times \mathcal{C}$, where $\mathcal{B}$ is the σ-field of Borel subsets of Ω.

THEOREM G. Assume that $\underset{\sim}{a}$ is μ-integrable, and that

(31.1) u is Borel-measurable;

(31.2) $u(x, s) = o(\|x\|)$ as $\|x\| \to \infty$, integrably in s;

(31.3) for each fixed s, u is continuous on Ω, and for each j, $\partial u(x, s)/\partial x^j$ exists and is continuous at each $x \in \Omega$ for which* $x^j > 0$; and

(31.4) $\underset{\sim}{a}(s) > 0$ for all s.

Then v (see (30.1)) is well-defined** and is in pNA, and the core of v consists of a single payoff vector, which coincides with the value φv.

Theorem G will be proved in Sec. 40.

Though it is common enough in economics, condition (31.4)--total positivity of initial resources--has a certain slightly restrictive, unintuitive flavor, and it would be nice if we could dispense with it. Two senses in which this can in fact be done will now be discussed.*** The first is to demand that there be only a finite number of different

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*I.e., whenever the two-sided partial derivative is defined.

**I.e., the maximum is attained for all $S \in \mathcal{C}$.

***For a third sense, see Proposition 33.2.

utility functions for the members of I. Specifically, let us say that u is <u>of finite type</u> if there is a finite set H of functions on Ω such that each of the functions $u(\cdot, s)$ is in H. (Note that this still allows all of the initial bundles $g(s)$ to be different.) Then we have

<u>PROPOSITION 31.5</u>. <u>Theorem G continues to hold if</u> (31.4) <u>is replaced by</u>

(31.6) u <u>is of finite type</u>.

Proposition 31.5 will be proved in Section 39.

The other sense in which (31.4) can be dispensed with is illustrated by the following proposition:

<u>PROPOSITION 31.7</u>. <u>Let</u> u <u>satisfy</u> (31.1), (31.2), <u>and</u> (31.3). <u>Then</u> v <u>is well defined, the asymptotic value of</u> v <u>exists, and the core of</u> v <u>consists of a single payoff vector, which coincides with the asymptotic value</u>.

Proposition 31.7 will be proved in Section 41. The proof depends on the "diagonal property" discussed in Section 19 (Part II). In fact, we will derive Proposition 31.7 from a more general proposition (Proposition 41.2), which is stated in terms of concepts related to the diagonal property.

Since we know that also the mixing value enjoys the diagonal property (Proposition 19.3), the question arises whether Proposition 31.8 could not be proved for the mixing value as well as the asymptotic value. We do not know the answer to this question, but the reader will find it discussed in Section 42. Other possibilities for extensions of the results stated here will also be discussed in Section 42.

32. THE COMPETITIVE EQUILIBRIUM

Let μ, u, $\underset{\sim}{a}$, and v be as in Sec. 30. An allocation is an integrable function $\underset{\sim}{x}$ from I to Ω such that

$$\int \underset{\sim}{x} = \int \underset{\sim}{a}.$$

A monetary competitive equilibrium (m.c.e.) is a pair $(\underset{\sim}{x}, p)$, where $\underset{\sim}{x}$ is an allocation and $p \in \Omega$, such that for all $s \in I$, $u(x, s) - p\cdot(x - \underset{\sim}{a}(s))$ attains its maximum (over $x \in \Omega$) at $x = \underset{\sim}{x}(s)$. The function on I whose value at s is $u(\underset{\sim}{x}(s), s) - p\cdot(\underset{\sim}{x}(s) - \underset{\sim}{a}(s))$ is called the competitive payoff density; its indefinite integral* (w.r.t. μ) is called the competitive payoff distribution; and p is the vector of competitive prices. (All three definitions are, of course, with respect to a given m.c.e. $(\underset{\sim}{x}, p)$.)

Intuitively, the vector p is a price vector. Thus, $p\cdot(\underset{\sim}{x}(s) - \underset{\sim}{a}(s))\mu(ds)$ represents the amount that the player** ds must pay in order to buy the bundle $\underset{\sim}{x}(s)\mu(ds)$, over and above the amount that he gets by selling his initial bundle $\underset{\sim}{a}(s)\mu(ds)$. This amount must be subtracted from $u(\underset{\sim}{x}(s), s)\mu(ds)$ in order to yield the net "income" of ds, and it is this income that ds wishes to maximize. If p is such that when

*If g is an integrable function on I, the indefinite integral of g is the measure ν defined by $\nu(S) = \int_S g d\mu$.

**Producer or trader, according to which interpretation is being used.

all players maximize in this way, the total demand $\int \underset{\sim}{x}$ equals the total supply $\int \underset{\sim}{a}$, then the economy is in equilibrium. Note that in the monetary interpretation, the total excess demand for money at such a point--namely, $\int p\cdot(\underset{\sim}{x} - \underset{\sim}{a})$--also vanishes.

We shall distinguish the concept just defined from the usual Walrasian concept of competitive equilibrium--as used, say, in $[A_1]$--by calling the latter a <u>barter competitive equilibrium</u> (b.c.e.).* To relate the two concepts, let us consider the monetary exchange economy interpretation of our game, namely, a market in which there are $n + 1$ goods $0, 1, \ldots, n$, the 0-th good being money. A b.c.e. in such a market takes the form of a pair $((\underset{\sim}{x}^0, \underset{\sim}{x}), (p^0, p))$, and we may assume w.l.o.g. that $p^0 = 1$. It is then easily verified that such a pair is a b.c.e. if and only if $(\underset{\sim}{x}, p)$ is an m.c.e. and for all s, $\underset{\sim}{x}^0(s) = p\cdot(\underset{\sim}{a}(s) - \underset{\sim}{x}(s))$. The total utility of the trader ds at this b.c.e. is then seen to be exactly

$$(u(\underset{\sim}{x}(s), s) - p\cdot(\underset{\sim}{x}(s) - \underset{\sim}{a}(s)))\mu(ds).$$

*The b.c.e. will not be formally defined here; the interested reader is referred to $[A_1]$ (for markets with a continuum of traders) or [D5] (for finite economies). Some familiarity with the concept of a b.c.e. is needed in certain parts of this section, e.g., in the proof of Proposition 32.5. It is however not needed in most of this section, e.g., for Propositions 32.1, 32.2, or 32.3 or their proofs. Neither is it used in the subsequent sections of this paper.

Note that although a b.c.e. remains a b.c.e. when the prices are multiplied by a positive constant, this is not the case for an m.c.e.; there the prices have already been normalized, so to speak, by the requirement that the price of money be 1.

PROPOSITION 32.1. Let u be Borel measurable, and let $\int \underset{\sim}{a} > 0$. Then an integrable $\underset{\sim}{x}$ maximizes $\int u(\underset{\sim}{x}(s), s)d\mu(s)$ subject to $\int \underset{\sim}{x} = \int \underset{\sim}{a}$ and $\underset{\sim}{x}(s) \in \Omega$ if and only if there is a p such that $(\underset{\sim}{x}, p)$ is a monetary competitive equilibrium.

This is essentially the content of Theorem 5.1 of [A-P] (cf. Proposition 36.4); it may be considered a form of the Kuhn-Tucker theorem [K-T] in an infinite dimensional space. The proposition says that any allocation $\underset{\sim}{x}$ for which $v(I)$ is attained (see (30.1)) is competitive, if the appropriate side payments $p \cdot (\underset{\sim}{x}(s) - \underset{\sim}{a}(s))$ are made. As for the prices p, when u is differentiable, then

$$p^i = [\partial u/\partial x^i]_{x=\underset{\sim}{x}(s)}$$

for all s such that $\underset{\sim}{x}^i(s) > 0$ (cf. (32.11)). Thus in the production interpretation, p^i is the marginal product of the i-th commodity at equilibrium, and in the market interpretation, it is the marginal utility (in both cases when there is some of the i-th commodity present at equilibrium).

<u>PROPOSITION 32.2</u>. <u>Assume</u> (31.1), (31.2), <u>and</u> $\int \underset{\sim}{a} > 0$. <u>Then there is an m.c.e</u>.

<u>Proof</u>. The main theorem of [A-P] asserts that under the conditions we have assumed,* the maximum in the definition of v is attained (cf. Proposition 36.1). The result then follows from Proposition 32.1. This completes the proof of Proposition 32.2.

Without (31.1) and (31.2), there may be no m.c.e.; see Section 33.

We now wish to discuss how the competitive equilibrium is related to the core and the value. In an ordinary Walrasian exchange economy** with a continuum of traders, it is known that the core coincides with the set of (barter) competitive allocations*** [A_1]. It is therefore reasonable to conjecture that a similar situation holds for m.c.e.'s. This is in fact the case; indeed we have

<u>PROPOSITION 32.3</u>. <u>Assume</u> (31.1), (31.2), (31.3), <u>and</u> $\int \underset{\sim}{a} > 0$. <u>Then there is a unique (monetary) competitive payoff distribution</u>,

*And even slightly weaker conditions.

**I.e., a "market without side payments".

***A <u>competitive allocation</u> in a barter economy is an allocation $\underset{\sim}{x}$ for which there exists a price vector p such that $(\underset{\sim}{x}, \tilde{p})$ is a b.c.e.

<u>which coincides with the unique point* in the core of</u> v, <u>and so also with the asymptotic value</u> φv.

<u>Remark</u>. Note that we are not asserting that the m.c.e. is unique. What <u>is</u> being asserted is that there is at least one m.c.e., and that if $(\underset{\sim}{x}, p)$ is any m.c.e., then

$$\int_S (u(\underset{\sim}{x}(s), s) - p \cdot (\underset{\sim}{x}(s) - \underset{\sim}{a}(s)))d\mu = (\varphi v)(S)$$

for all $S \in \mathcal{C}$.

<u>Proof</u>. By Proposition 32.2, there is an m.c.e. $(\underset{\sim}{x}, p)$. Let ν be the corresponding competitive payoff distribution. Since $\underset{\sim}{x}$ is an allocation it follows that

$$\nu(I) = v(I). \tag{32.4}$$

Next, if S is any coalition, let v(S) be attained at $\underset{\sim}{y}$, i.e.,

$$v(S) = \int_S u(\underset{\sim}{y}(s), s)ds, \quad \int_S \underset{\sim}{y} = \int_S \underset{\sim}{a}, \quad \text{and } \underset{\sim}{y}(s) \geqq 0 \quad \text{for all } s.$$

Then by the definition of m.c.e.,

*See Proposition 31.7.

$$u(\underset{\sim}{x}(s), s) - p\cdot(\underset{\sim}{x}(s) - \underset{\sim}{a}(s)) \geq u(\underset{\sim}{y}(s), s) - p\cdot(\underset{\sim}{y}(s) - \underset{\sim}{a}(s)).$$

Integrating this over S, we obtain

$$\nu(S) \geq v(S) - p\cdot\int_S(\underset{\sim}{y} - \underset{\sim}{a}) = v(S);$$

together with (32.4), this shows that ν is in the core. But by Proposition 31.7, the core contains a unique point, namely the asymptotic value; so the proof of Proposition 32.3 is complete.

In the above proof, we made use of the fact that there is only one point in the core in order to establish the equivalence between the core and the set of all competitive payoff distributions. The proof of uniqueness for the core, in turn, is intimately bound up with value considerations and with the differentiability of u. But the equivalence between the core and the set of competitive allocations is a much more general phenomenon, which does not depend on differentiability, is not directly connected with value considerations, and in fact continues to hold even when the core has many members. It is therefore of some interest to establish this equivalence under conditions that are more general than those of Proposition 32.3 even though there is no direct connection betwen this and the value.

PROPOSITION 32.5. Assume that u is continuous in x for each fixed s and is Borel measurable, that v is well-defined,* and that $\int \underset{\sim}{a} > 0$. Then the core of v coincides with the set of (monetary) competitive payoff distributions.

Proof. The idea of the proof is to introduce "money" explicitly, as in the monetary exchange interpretation of our economy. We then get an $(n + 1)$-good market whose b.c.e.'s are in 1 - 1 correspondence with the m.c.e.'s of the original economy, and whose core corresponds** to the core of v. We may now apply the "equivalence theorem" for barter economies (see, e.g., [A_1]), according to which the core of such an economy coincides with the set of all barter competitive allocations (b.c.a.'s)--i.e., allocations associated with some b.c.e. Since the core of the $(n+1)$-good barter economy corresponds to the core of the original n-good monetary economy, we may deduce the equivalence in the original monetary economy.

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*I.e., that for each S, the maximum in the definition of $v(S)$ is attained; (31.2) is a sufficient condition for this, but it is not necessary. Incidentally, all that is needed for this proposition is that the max in the definition of $v(I)$ be achieved; if for the other S, $v(S)$ is defined to be the sup rather than the max, the proposition remains true.

**It is in establishing the correspondence between the cores that one uses the assumption that the max in the definition of $v(I)$ is achieved.

Unfortunately, we are unable to use the equivalence theorem of [A_1] for this purpose, for the following reason: In the (n+1)-good barter economy, money is available in negative as well as nonnegative quantities, whereas all other goods are available in nonnegative quantities only. Therefore the space of all commodity bundles is not the nonnegative orthant of E^{n+1}, but rather $E^1 \times \Omega$, where E^1 is the entire real line and Ω is the nonnegative orthant of E^n. But the equivalence theorem of [A_1] is stated only for the case in which the space of commodity bundles is precisely the nonnegative orthant.*

Fortunately, a more general form of the equivalence theorem is available [Hi]; in this theorem, for each $s \in I$ there is a <u>consumption set</u> $X(s)$, which is only assumed to be a convex subset of E^n (rather than $X(s) = \Omega$, as in [A_1]). To describe the result, we must recall the concept of a <u>quasi-competitive allocation</u> (in a barter economy), due to Debreu [D_6]; it is an allocation $\underset{\sim}{x}$ for which there exists a price vector p, such that for almost all s, either $\underset{\sim}{x}(s)$ is maximal in the budget set of s, or $p \cdot \underset{\sim}{x}(s)$ is the minimum of $p \cdot x$ as x ranges over the consumption set $X(s)$. We then have, under appropriate conditions, that

*Though we believe that the proof in [A_1] would go through in the case considered here without any difficulty.

(32.6) the core coincides with the set of quasi-competitive allocations

[Hi, Theorem 2, p. 448]. In our application, where $X(s) = E^1 \times \Omega$ for all s, we now show

(32.7) every quasi-competitive allocation is competitive.

Indeed, if the price of money is not 0, then $p \cdot x$ takes arbitrarily small values in $E^1 \times \Omega$, so the minimum cannot be attained at all, and (32.7) follows immediately. If the price of money is 0, then no $\underset{\sim}{x}(s)$ can be maximal in the budget set of s, because by adding some money to $\underset{\sim}{x}(s)$ one gets a more preferred* bundle while still remaining in the budget set. So

(32.8) $$p \cdot \underset{\sim}{x}(s) = \min (p \cdot x : x \in E^1 \times \Omega)$$

for all s. Now since the price of money vanishes there must be at least one ordinary good with a nonvanishing price, say the good indexed by 1. If all prices are non-negative, then it follows that $p^1 > 0$; since $\int \underset{\sim}{x}^1 = \int \underset{\sim}{a}^1 > 0$, there is an s with $\underset{\sim}{x}^1(s) > 0$, so

*Recall that $(x^0, x) \succ (y^0, y) \Leftrightarrow u(x) + x^0 > u(y) + y^0$.

$$p \cdot \underset{\sim}{x}(s) \geq p^1 \cdot \underset{\sim}{x}^1(s) > 0 = \min (p \cdot x : x \in E^1 \times \Omega),$$

in contradiction to (32.8). If at least one price is negative, then the minimum on the right side of (32.8) cannot be attained, since the infimum is $-\infty$; so again (32.8) is contradicted. Thus (32.7) is established, and so from (32.6) it follows that the core coincides with the set of competitive allocations. As for the "appropriate conditions" needed for Hildenbrand's theorem, these include continuity and measurability of the preferences and a local nonsatiation condition, and are all easily verified here. Hildenbrand's set-up also includes production sets for all coalitions, but this can be dispensed with here; we simply let the production sets coincide with the nonpositive orthant.

Summing up, we have shown that Hildenbrand's result yields the Equivalence Theorem for the barter economy corresponding to our original monetary economy. The remainder of the proof can now be completed as outlined above. This completes the proof of Proposition 32.5.

In the proofs of the theorems stated in Section 31, the uniqueness of the core* is established via value considerations, using Theorem F; strong use is thereby made

*This is a rather loose, though convenient, method of expression. Strictly speaking, it is the point in the core that is unique; the core itself, as a set, is trivially unique, in any game.

of the "differentiability" of v--i.e., the existence of $\partial v^*(t, S)$--along the diagonal, and this in turn depends on the differentiability of u. Proposition 32.5 gives us an opportunity to establish the uniqueness of the core in a different manner, by proving the uniqueness of the competitive payoff distribution. As may be expected, this too depends on the differentiability of u.

Let u satisfy (31.1) and (31.3), and let $\int \underset{\sim}{a} > 0$. Let $(\underset{\sim}{x}, p)$ be an m.c.e. From the definition of m.c.e. it then follows that for all $x \in \Omega$,

$$u(\underset{\sim}{x}(s), s) - p\cdot\underset{\sim}{x}(s) \geq u(x, s) - p\cdot x,$$

whence

$$u(x, s) - u(\underset{\sim}{x}(s), s) \leq p\cdot(x - \underset{\sim}{x}(s)). \tag{32.9}$$

Setting $x = \underset{\sim}{x}(s) + \delta e_j$ for a given j and letting $\delta \to 0+$, we deduce

$$[\partial u/\partial x^j]_{x=\underset{\sim}{x}(s)} \leq p^j. \tag{32.10}$$

If, moreover, $\underset{\sim}{x}^j(s) > 0$, then we may let $\delta \to 0-$, obtaining the inequality opposite to (32.10); together, they yield

$$(32.11)\quad [\partial u/\partial x^j]_{x=\underset{\sim}{x}(s)} = p^j \quad \text{whenever } \underset{\sim}{x}^j(s) > 0.$$

Since $\int \underset{\sim}{x}^j = \int \underset{\sim}{a}^j > 0$, there must be an s such that $\underset{\sim}{x}^j(s) > 0$. Thus the competitive prices are uniquely determined. But then there can be at most one competitive payoff density, namely

$$(32.12)\qquad \max\, (u(x,\, s) - p\cdot(x - \underset{\sim}{a}(s))),$$

and so at most one competitive payoff distribution.

If, moreover, u also satisfies (31.2), then by Proposition 32.2, there is an m.c.e., i.e., the max in (32.12) is attained. Thus we have provided an alternative proof, which does not depend on the value concept, of all but the last clause of Proposition 32.3.

Theorem F provides a direct connection between the value and the core, and what we have just said provides the corresponding connection between the m.c.e. and the core. To complete the triangle, we now demonstrate directly how the value is connected with the m.c.e., without considering the core. Unlike the previous demonstrations, though, this demonstration will have a heuristic rather than a strictly rigorous nature.*

*Though there were gaps in the previous demonstrations, they are relatively easily filled in. The gaps in the present argument are more serious.

If $\underset{\sim}{x}$ is a measurable function from I to Ω, we will find it convenient slightly to abuse our notation by writing $u(\underset{\sim}{x})$ for the function on I whose value at s is $u(\underset{\sim}{x}(s), s)$.

Assume that it has been established that $v \in pNA$. If f is an ideal set (see Part III), it then seems reasonable to suppose that

$$(32.13)\quad v^*(f) = \max\{\int u(\underset{\sim}{x})f : \int \underset{\sim}{x}f = \int \underset{\sim}{a}f \text{ and } \underset{\sim}{x}(s) \geq 0 \text{ for all } s\}.$$

Assuming (32.13), let us, for given $S \in \mathcal{C}$ and $t \in (0,1)$, calculate the expression

$$\partial v^*(t, S) = \lim_{\tau \to 0} \frac{v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I)}{\tau}.$$

From (32.13) it follows that

$$v^*(t\chi_I) = tv(I) = \int tu(\underset{\sim}{x}),$$

where $\underset{\sim}{x}$ is the allocation at which v(I) is achieved. Now let $v^*(t\chi_I + \tau\chi_S)$ be achieved at $\underset{\sim}{y}$. Then for sufficiently small τ, we have

$$(32.14)\quad v^*(t\chi_I + \tau\chi_I) = \int(u(\underset{\sim}{y}) - u(\underset{\sim}{x}))(t\chi_I + \tau\chi_S) + \tau\int_S u(\underset{\sim}{x})$$

$$\leq \int[p\cdot(\underset{\sim}{y} - \underset{\sim}{x})](t\chi_I + \tau\chi_S) + \tau\int_S u(\underset{\sim}{x})$$

$$= p\cdot\int(\underset{\sim}{a} - \underset{\sim}{x})(t\chi_I + \tau\chi_S) + \tau\int_S u(\underset{\sim}{x})$$

$$= tp\cdot\int(\underset{\sim}{a} - \underset{\sim}{x}) + \tau p\cdot\int_S(\underset{\sim}{a} - \underset{\sim}{x}) + \tau\int_S u(\underset{\sim}{x})$$

$$= 0 + \tau\int_S(u(\underset{\sim}{x}) - p\cdot(\underset{\sim}{x} - \underset{\sim}{a})),$$

and hence

$$(32.15)\quad \partial v^*(t, S) \leq \int_S(u(\underset{\sim}{x}) - p\cdot(\underset{\sim}{x} - \underset{\sim}{a})).$$

We can, however, say more, namely that equality holds in (32.15). To show this, it is only necessary to point to a $\underset{\sim}{y}$ such that

$$(32.16)\quad \int(t\chi_I + \tau\chi_S)\underset{\sim}{y} = \int(t\chi_I + \tau\chi_S)\underset{\sim}{a} \text{ and } \underset{\sim}{y}(s) \geq 0 \text{ for all } s,$$

for which the inequality in (32.14) becomes an equality up to a term that is $o(\tau)$. Now it is always possible to find a $\underset{\sim}{y}$ satisfying (32.16) which will have the property that $\underset{\sim}{y}^j(s) = 0$ whenever $\underset{\sim}{x}^j(s) = 0$; and moreover, such that $\underset{\sim}{y}^j(s) = \underset{\sim}{x}^j(s) + K^j(s)\tau$ for all s, where $K^j(s)$ is a constant that depends on j and on whether s is or is not in S, but otherwise does not depend on s or on τ. From this and (32.11) it follows that equality holds in (32.14), and

hence in (32.15). But then it follows from Theorem E that the value φv coincides with the competitive payoff distribution.

The two major gaps in this argument are the unproven assumptions that $v \in pNA$ and that v^* is given by (32.13). Given $v \in pNA$, (32.13) is probably not too hard to prove, e.g. by the theorems of Section 25. But to prove $v \in pNA$, say from the assumptions of Theorem G, is a serious bit of work. Indeed, it is precisely this that constitutes the most difficult part of the proof of Theorem G, and it will require every bit of Sections 35 through 40 before it is done.

We repeat, though, that our proof of Theorem G will not depend on the above argument (nor will it depend explicitly on the m.c.e. at all); rather, it will use Theorem F, i.e., it will depend on core considerations only. The above arguments were only given to shed light on the relations between core, value, and competitive equilibrium, from several different viewpoints. From the point of view of this paper, the m.c.e. is strictly speaking not needed at all; and if it <u>is</u> introduced, it is most directly related to the core and the value via the first proof of Proposition 32.3 given above.

33. EXAMPLES

In all the numbered examples of this section,* I will be the unit interval $[0,1]$, $\mathcal{C}$ the Borel σ-field $\mathcal{B}$, and μ Lebesque measure λ.

A. The Case $n = 1$

In the production interpretation, $n = 1$ means that the finished good is produced from only one kind of raw material, though the efficiency of production of the various traders ds--i.e., the production functions $u(\cdot,s)\mu(ds)$--may be different.** In the exchange interpretation, we are dealing with a market in which only one kind of good is being bought and sold (for money), the demand for (and supply of) this one good being created by the different utility functions $u(\cdot,s)\mu(ds)$ that the traders ds have for the good. Conceptually and computationally, this case is somewhat easier to deal with than the one of general n. Yet it is far from trivial, and its analysis involves most of the basic ideas that are met with in the general case.

Example 33.1. Let $n = 1$, and for all s, let

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*Examples 33.1, 33.3, 33.6, 33.9, 33.11, 33.12, 33.13.

**It may even happen that one trader produces more efficiently than another at a certain level, whereas the other produces more efficiently than the first at a different level.

$$u(x, s) = \sqrt{x + s} - \sqrt{s}$$

(see Fig. 1) and

$$\underset{\sim}{a}(s) = 1/32.$$

This market satisfies (31.1), (31.2), (31.3), and (31.4). Therefore, from Theorem G and Proposition 32.3, it follows that $v \in pNA$, the core of v and the m.c.e. are unique, and both coincide with the value. It is easiest to compute the m.c.e., making use of (32.10) and (32.11). The idea of the computation is that the higher the price p is, the smaller will be the total demand for the good x; we must find a price at which the total demand exactly matches the total supply $\int \underset{\sim}{a} = 1/32$. Suppose then that the price is p; let $\underset{\sim}{x}(s)\mu(ds)$ be the demand of ds. Then by (32.10), if $\underset{\sim}{x}(s) > 0$, we have

$$p = [\partial u/\partial x]_{x=\underset{\sim}{x}(s)} = \frac{1}{2\sqrt{\underset{\sim}{x}(s) + s}};$$

hence

$$\underset{\sim}{x}(s) = \frac{1}{4p^2} - s \quad \text{and} \quad u(\underset{\sim}{x}(s), s) = \frac{1}{2p} - \sqrt{s}.$$

By (32.11), if $\underset{\sim}{x}(s) = 0$, then

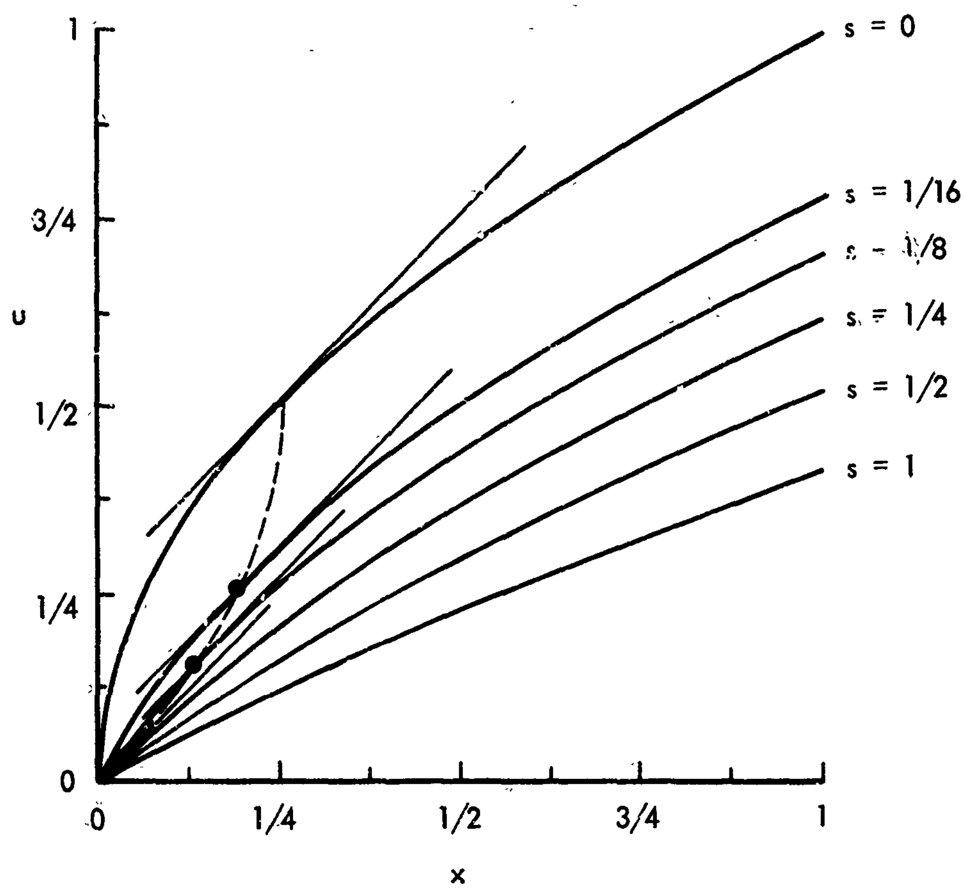


Fig. 1 — The function u(s, x) for Example 33.1

$$\frac{1}{2\sqrt{s}} = [\partial u/\partial x]_{x=0} \leqq p;$$

hence,

$$\frac{1}{4p^2} - s \leqq 0.$$

Thus, we conclude that in any case

$$\underset{\sim}{x}(s) = \max\ (0, \frac{1}{4p^2} - s).$$

Hence,

$$\frac{1}{32} = \int \underset{\sim}{a} = \int \underset{\sim}{x} = \int_0^{\frac{1}{4p^2}} (\frac{1}{4p^2} - s)ds$$

$$= \int_0^{\frac{1}{4p^2}} sds = [\frac{s^2}{2}]_0^{\frac{1}{4p^2}} = \frac{1}{32p^4}.$$

Hence $p = 1$, and it follows that

$$v(I) = \int_0^1 u(\underset{\sim}{x}(s), s)ds = \int_0^{\frac{1}{4}} (\frac{1}{2} - \sqrt{s})ds = \frac{1}{2}\cdot\frac{1}{4} - \frac{2}{3}(\frac{1}{4})^{3/2} = \frac{1}{24}.$$

The competitive payoff density is

$$[u(\underset{\sim}{x}(s), s) - p\cdot\underset{\sim}{x}(s)] + p\cdot\underset{\sim}{a}(s);$$

when $s \geqq \frac{1}{4}$ this consists simply of $p\cdot\underset{\sim}{a}(s) = \frac{1}{32}$. When

$s \leq \frac{1}{4}$, we have in addition to this amount, the amount

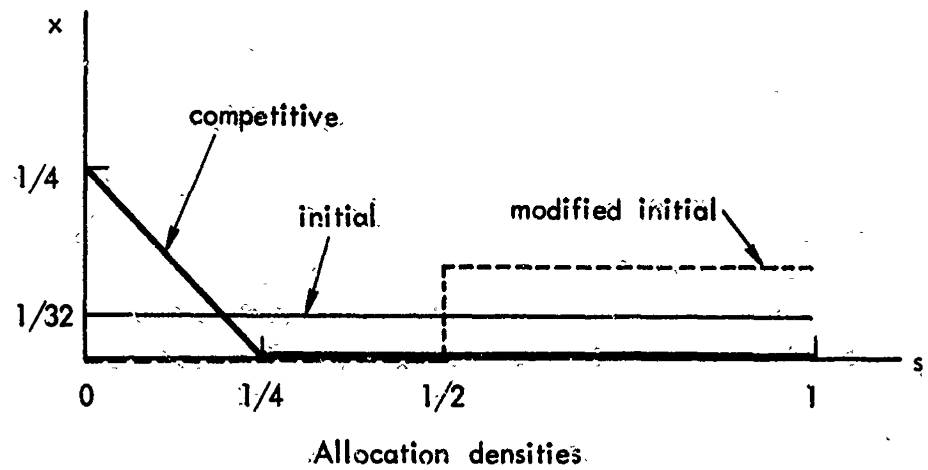
$$\frac{1}{2} - \sqrt{s} - (\frac{1}{4} - s) = \frac{1}{4} + s - \sqrt{s} = (\frac{1}{2} - \sqrt{s})^2,$$

which ranges from 0 at $s = \frac{1}{4}$ to $\frac{1}{4}$ at $s = 0$. The situation is pictured in Fig. 2 (solid lines). The slopes of the u-curves at $x = \underset{\sim}{x}(s)$ (dashed line in Fig. 1) are all equal to the competitive price of 1 when $\underset{\sim}{x}(s) > 0$, but when $\underset{\sim}{x}(s) = 0$ the tangent at 0 may have a slope smaller than 1.

Note that the competitive payoff density may be thought of as consisting of two parts, namely $p \cdot \underset{\sim}{a}(s)$ and $u(\underset{\sim}{x}(s), s) - p \cdot \underset{\sim}{x}(s)$. In the production interpretation these two parts may be thought of as follows: the first part is compensation to ds in his role as supplier, and is always divided among the players in proportion* to $\underset{\sim}{a}(s)$. The second part is attributable to his role as producer, i.e., to his u-function $u(\cdot, s)$, and does not depend in any way on his initial bundle** $\underset{\sim}{a}(s)$.

*This needs no interpretation when $n = 1$. When $n > 1$, it means in proportion to $p \cdot a(s)$. However, even when $n > 1$, if one trader's initial bundle density is exactly twice that of another--in the vectorial sense--then that part of his payoff density attributable to his role as a supplier will also be twice that of the other trader.

**It is interesting to remark that this division into two parts with the above properties is far from trivial if one looks at the payoff from the value or core point of view.



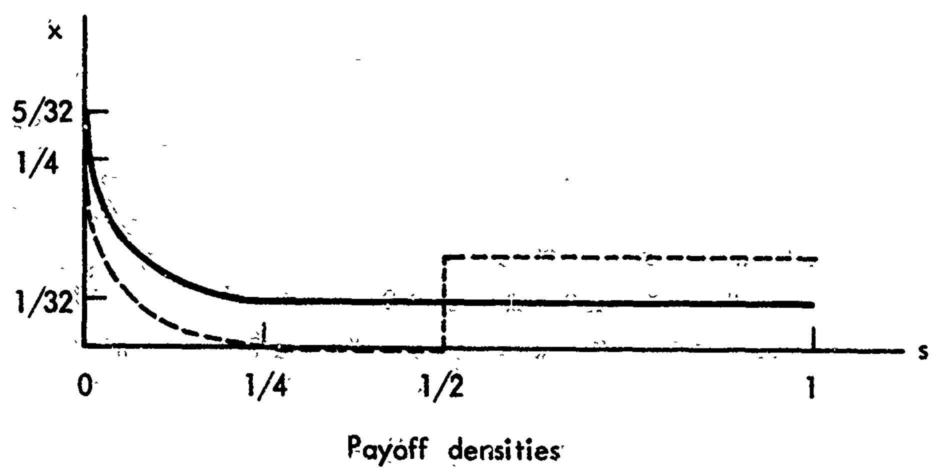


Fig.2 — Competitive solution for Example 33.1

If, for example, we redefine the initial bundle distribution here by

$$\underset{\sim}{a}(s) = \begin{cases} 0, & 0 \leq s \leq \frac{1}{2} \\ \frac{1}{16}, & \frac{1}{2} < s \leq 1, \end{cases}$$

then our calculation will remain essentially unchanged, the only difference occurring in the payoff density. This is illustrated by the dashed lines in Fig. 2. The players ds for $s \in [0,\frac{1}{4}]$ will act as producers, and will obtain a payoff density of $\frac{1}{4} + s - \sqrt{s}$; they will obtain nothing as suppliers, since they have no initial bundles. For $s \in [\frac{1}{4}, \frac{1}{2}]$, the players have no initial supplies, neither are they sufficiently efficient to produce; therefore, their payoff is 0. The remaining players (those between $\frac{1}{2}$ and 1) get a payoff density of 1/16, in proportion to their initial holdings; but they are not sufficiently efficient as producers to act in this capacity. One might say that they sell their initial holdings to the efficient producers between 0 and $\frac{1}{4}$ in return for a promise of manufactured goods.

This second version of the example does not satisfy (31.4), and therefore we cannot deduce from Theorem G that $v \in pNA$. However, we have

<u>PROPOSITION 33.2</u>. <u>In Theorem</u> G, (31.4) <u>may be replaced by the assumption that</u> $n = 1$.

This proposition will be proved in Section 40, together* with Theorem G. Readers who have followed the preceding example will realize that even when $n = 1$, the assumption $\underset{\sim}{a}(s) > 0$ is by no means of a trivial nature;** players for whom $\underset{\sim}{a}(s) = 0$ may still have considerable significance as producers (in the production interpretation), even though they supply none of the initial good. Thus Proposition 33.2 is by no means an easy consequence of Theorem G.

B. The Finite Type Case

Suppose that u satisfies (31.1), (31.2), and (31.3), and moreover it is of finite type; that is, there are finitely many functions $f_1, \ldots, f_k$ on Ω such that each of the functions $u(\cdot,s)$ is one of the f_i. Define a k-dimensional vector η of measures on I by

$$\eta_i(S) = \mu\{s : u(\cdot, s) = f_i\},$$

and an n-dimensional vector ζ of measures on I by

$$\zeta(S) = \int_S \underset{\sim}{a}.$$

*More precisely, a common generalization (Proposition 40.26) of Theorem G and Proposition 33.2 will be proved.

**As it would be (because of $n = 1$) in most discussions of a barter economy. When money is introduced explicitly, a monetary economy with $n = 1$ becomes a barter economy with $n = 2$ (cf. the proof of Proposition 32.5).

Let $\nu = (\eta, \zeta)$. Then v is a function* of the n + k dimensional vector ν, say

$$v = g \circ \nu.$$

Let us calcualte g in a specific example.

<u>Example 33.3</u>. Let n = 1 and for all s, let

$$u(x, s) = \sqrt{x + 1} - 1$$

and

$$\underset{\sim}{a}(s) = 8s.$$

In this case, both η (which = λ) and ζ are one-dimensional, so $\nu = (\eta, \zeta)$ is two-dimensional. The range R of ν is depicted in Fig. 3; note that it is not symmetric around the diagonal, but rather (as always) around the center of the diagonal. It may be seen that $v = g \circ \nu$, where

$$g(y, z) = y[\sqrt{\frac{z}{y} + 1} - 1] = \sqrt{y(y + z)} - y.$$

*For a detailed discussion, see Section 39, in particular formulas (39.7) and (39.18).

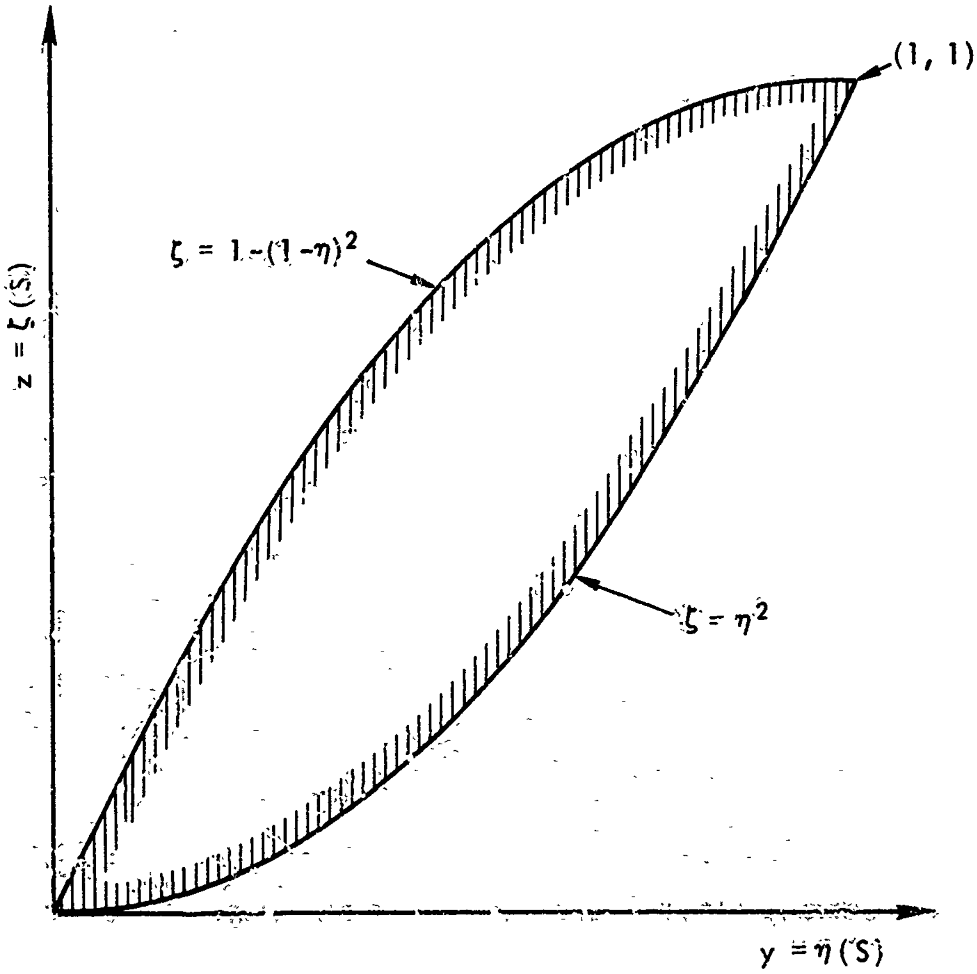


Fig.3 — The range of $\nu = (\eta, \zeta)$ in Example 33.3

We would now like to apply Theorem B to deduce* that $v \in pNA$ and to obtain the value φv. Unfortunately, this is impossible, because the conditions of Theorem B fail; g is not continuously differentiable on the range R. Indeed, we have

$$\partial g/\partial z = \frac{1}{2}\sqrt{y/(y+z)}. \tag{33.4}$$

If we let $(y, z) \to 0$ along the diagonal, then $\partial g/\partial z \to 1/2\sqrt{2}$, whereas if we let $(y, z) \to 0$ along the bottom boundary of R, then $\partial g/\partial z \to \frac{1}{2}$. Hence, $\partial g/\partial z$ cannot be extended to all of R so that it will be continuous at 0.

Though Theorem B is not applicable, Proposition 9.17 is, and we apply it to deduce that $v \in pNA$. To calculate the value φv, we would like to use the "diagonal formula" (3.2). Though we have not heretofore proved this under the conditions of Proposition 9.17, it does in fact hold under those conditions.** Using (33.4) and

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*Of course we know from Proposition 31.5 that $v \in pNA$; what we are investigating here is whether a simple proof can be obtained for this very simple special case.

**Probably the easiest way to establish this at this stage of the game is to use Theorem E. If one wants to restrict oneself to more elementary methods, it is not difficult to devise a proof using Proposition 9.7, and the fact that $f^{\circ} \circ \mu$ satisfies the conditions of Theorem B.

$$\frac{\partial g}{\partial y} = \frac{2y + z}{2\sqrt{y(y + z)}} - 1,$$

we thus obtain

$$(33.5)\quad (\varphi v)(S) = \eta(S)\int_0^1 \frac{\partial g}{\partial y}(t,\ t)dt + \zeta(S)\int_0^1 \frac{\partial g}{\partial z}(t,\ t)dt$$

$$= \left(\frac{3}{2\sqrt{2}} - 1\right)\eta(S) + \frac{1}{2\sqrt{2}}\zeta(S).$$

In the production interpretation, the first and second terms may be considered compensation to the members of S in their roles as producers and suppliers respectively (cf. the discussion of Example 33.1). As we shall see in Section 39, $g \circ \nu$ satisfies the conditions of Proposition 9.17 whenever u is of finite type, and so the value formula applies. This implies a decomposition of φv into a term involving η only (production) and a term involving ζ only (supply), so that for the finite type case we have a derivation of this phenomenon from value considerations as well.

In this example, v(S) is always achieved by $\underset{\sim}{x}(s) \equiv \zeta(S)$; in particular, v(I) is achieved by $\underset{\sim}{x}(s) \equiv 1$. Hence in the m.c.e., we have

$$p = [\partial u/\partial x]_{x=1} = \left[\frac{1}{2\sqrt{x+1}}\right]_{x=1} = \frac{1}{2\sqrt{2}}.$$

Hence, for all s, the competitive payoff density is given by

$$u(\underset{\sim}{x}(s), s) - p\cdot(\underset{\sim}{x}(s) - \underset{\sim}{a}(s)) = \sqrt{\underset{\sim}{x}(s) + 1} - 1 - \frac{1}{2\sqrt{2}}(\underset{\sim}{x}(s) - s)$$

$$= \sqrt{2} - 1 - \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} s$$

$$= \frac{3}{2\sqrt{2}} - 1 + \frac{1}{2\sqrt{2}} s;$$

hence the competitive payoff distribution is given by (33.5), which is as it should be. In the general finite type case as well, it may be seen by direct computations that the diagonal formula for the value yields the competitive payoff distributions (cf. Section 39, especially the material following (39.7)).

The main point of Example 33.3 was to show that Theorem B is not sufficient to deal even with the simplest* finite type cases, but that Proposition 9.17 is needed. In Section 39 we shall see that Proposition 9.17 is indeed sufficient to cover the general finite type case.

C. Differentiability

To show that condition (31.3) cannot be dispensed with, consider the following market:

Example 33.6. Let $n = 1$, and for all s, let

*We used $\sqrt{x + 1} - 1$ rather than simply $\sqrt{x}$ in order to show that even when u is differentiable on the n-dimensional nonnegative orthant, g may not be differentiable on the n+k-dimensional orthant, and Theorem B may not be applicable.

$$u(x, s) = \begin{cases} x, & \text{for } x \leqq 1 \\ \sqrt{x}, & \text{for } x \geqq 1, \end{cases}$$

and

$$\underset{\sim}{a}(s) = \begin{cases} 3/2, & \text{for } s \leqq \frac{1}{2}, \\ 1/2, & \text{for } s > \frac{1}{2}. \end{cases}$$

The function $u(x, s)$ is graphed in Fig. 4. It is not differentiable at 1; the left derivative is 1, and the right derivative is $\frac{1}{2}$. Thus we define $\underset{\sim}{x}(s) \equiv 1$, then $(\underset{\sim}{x}, p)$ is an m.c.e. whenever $\frac{1}{2} \leqq p \leqq 1$, because the line through (1, 1) with slope p supports the graph of u for those values of p. The competitive payoff density corresponding to a given value of p will therefore be

$$u(\underset{\sim}{x}(s), s) - p\cdot(\underset{\sim}{x}(s) - \underset{\sim}{a}(s)) = \begin{cases} 1 + \frac{1}{2}p, & \text{for } s \leqq \frac{1}{2} \\ 1 - \frac{1}{2}p, & \text{for } s > \frac{1}{2}. \end{cases}$$

The competitive payoff distribution is therefore given by

$$(33.7)\quad \xi_p(S) = (1 + \tfrac{1}{2}p)\lambda(S \cap [0,\tfrac{1}{2}]) + (1 - \tfrac{1}{2}p)\lambda(S \cap [\tfrac{1}{2},1]).$$

It follows from Proposition 32.5 that the core of the market is the set of all ξ_p, where p ranges from $\frac{1}{2}$ to 1. In particular, it consists of more than one point, so that the conclusion of Theorem G (and also that of Proposition 31.7) fails.

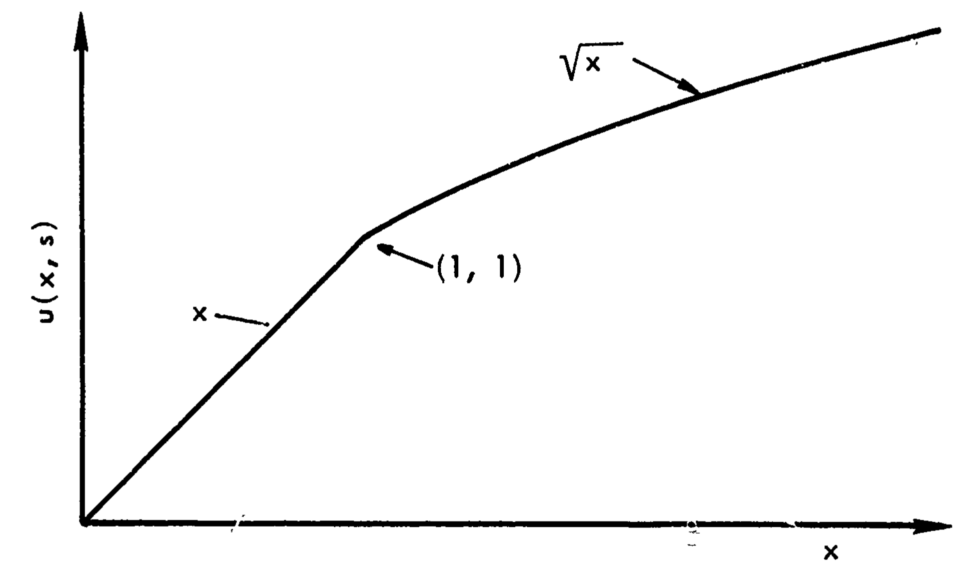

Fig.4 — The function u(s, x) for Example 33.6

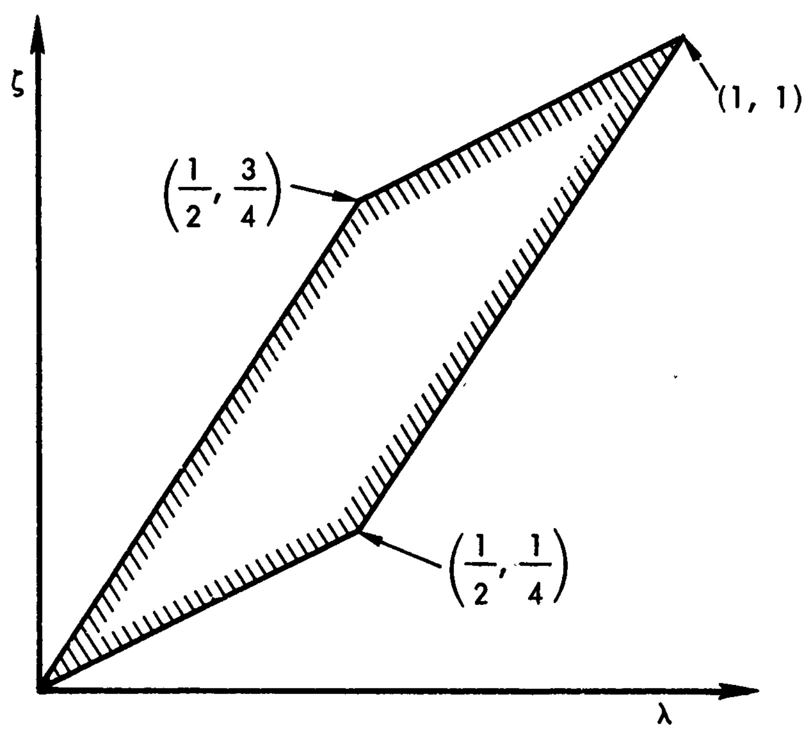

Fig.5 — The range of (λ, ζ) in Example 33.6

We can also calculate the core directly. Let $\zeta(S) = \int_S a$; the range of (λ, ζ) is depicted in Fig. 5. It may be verified that

$$(33.8) \qquad v = \min(\zeta, \sqrt{\zeta\lambda}).$$

Since $\frac{1}{2}(\zeta + \lambda) \geq \sqrt{\zeta\lambda}$ and $\zeta(I) = \lambda(I) = 1$, it follows that the core of v contains all convex combinations of the form $t\zeta + (1 - t)\lambda$, with $\frac{1}{2} \leq t \leq 1$; these are precisely the ξ_p of (33.7).

The reader will note the similarity between formulas (33.8) and (3.4); in neither case does v belong to pNA. Indeed, the proof that the v of (33.8) does not belong to pNA can be carried out along the same lines as the proof of Example 5.8 appearing at the end of Section 27. Rather than doing this in detail, though, we will present another nondifferentiable market more directly related to (3.4) and Example 5.8.

<u>Example 33.9</u>. Define a function f on the nonnegative half-line by

$$f(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ 2 - \frac{1}{x}, & \text{for } 1 \leq x \end{cases}$$

(see Fig. 6). Let n = 2, and for all s, let

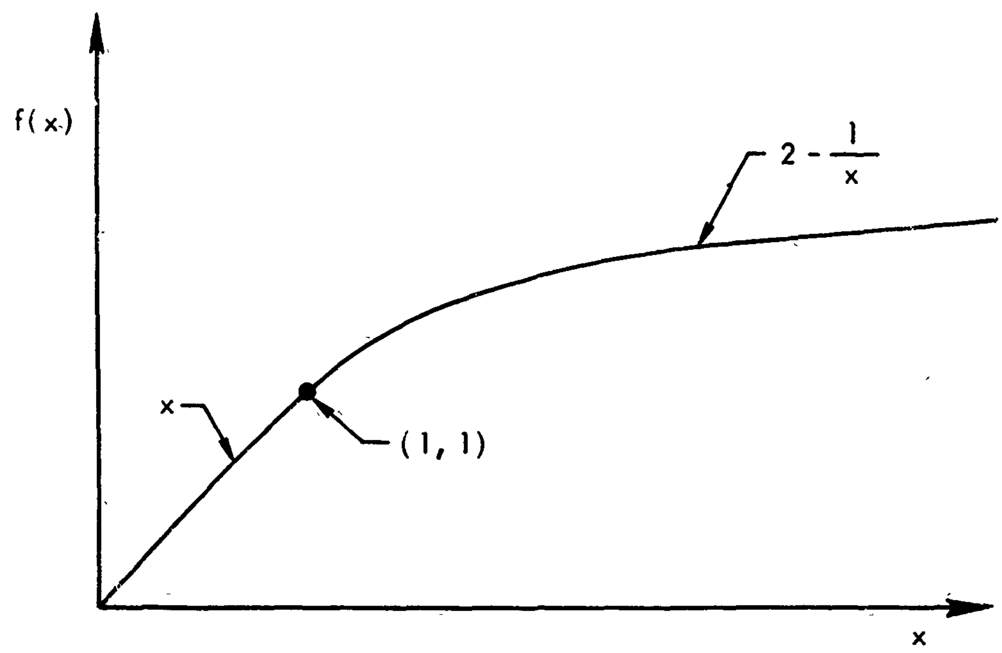

Fig.6 — The function f(x) in Example 33.9

$$u(x, s) = f(\min(x^1, x^2) + (x^1 + x^2)/10))$$

and

$$\underset{\sim}{a}(s) = \begin{cases} (\frac{3}{2}, \frac{1}{2}) & \text{for } 0 \leq s \leq \frac{1}{2} \\ (\frac{1}{2}, \frac{3}{2}) & \text{for } \frac{1}{2} < s \leq 1. \end{cases}$$

This is often called the "glove market"; it has the following (exchange) interpretation:* The commodities 1 and 2 are left and right gloves respectively. Individual gloves are next to useless, being useable only for the material in them (this accounts for the term $(x^1 + x^2)/10$, which is needed so that u be strictly increasing). Pairs of gloves, however, can be used as gloves. The utility for pairs of gloves and for material is bounded, being governed by the function f (this is needed to ensure that (31.2) is obeyed).

It is easy to see that v(I) is achieved for $\underset{\sim}{x}(s) \equiv (1, 1)$. The prices p must satisfy

$$p^1 \geq 0.1,\ p^2 \geq 0.1,\ p^1 + p^2 = 1.2;$$

otherwise, however, they are arbitrary. In other words, p may be any convex combination of (1.1, 0.1) and (0.1, 1.1). The set of all competitive payoff distributions--i.e., the core--may be easily calculated from this. Alternatively

*Cf. [S-S_3], pp. 342-347.

we may proceed as follows: Define $\zeta(S) = \int_S \underset{\sim}{a}$. Then

$$(33.10) \qquad v = \min(\zeta^1, \zeta^2) + \frac{1}{10}(\zeta^1 + \zeta^2),$$

and it is easily verified from this that any convex combination of $\zeta^1 + \frac{1}{10}(\zeta^1 + \zeta^2)$ and $\zeta^2 + \frac{1}{10}(\zeta^1 + \zeta^2)$ is in the core of v. But if we set

$$\lambda_1(S) = \lambda(S \cap [0, \tfrac{1}{2}])$$

$$\lambda_2(S) = \lambda(S \cap [\tfrac{1}{2}, 1]),$$

then from (33.10) it follows that

$$v = \frac{6}{10}(\zeta^1 + \zeta^2) + \frac{1}{2}|\zeta^1 - \zeta^2| = \frac{6}{5}\lambda + \frac{1}{2}|\lambda_1 - \lambda_2|,$$

and hence it follows immediately from Example 5.8 that $v \notin pNA$.

D. Achievement of the Max in the Definition of v

If condition (31.2) is not obeyed, the max in the definition of v(S) may not be achieved, even though the sup may be finite. The following example of this is from [A - P]:

Example 33.11. Let n = 1, and for all s, let

$$u(x, s) = xs$$

and

$$\underset{\sim}{a}(s) = 1.$$

In this case, the integral appearing in the definition of $v(I)$ is $\int s\underset{\sim}{x}(s)ds$, and this must be maximized subject to $\int \underset{\sim}{x} = 1$; the supremum in this case is 1, but it is not achieved.

Since $v(I)$ is not achieved, it follows from Proposition 32.1 that this economy has no m.c.e. Therefore, one cannot hope to extend Proposition 32.3, according to which the core, value, and competitive payoff distributions all coincide, to this situation. But possibly an extension of Theorem G could be proved, i.e., maybe we could show that if in the definition (31.1) of v we replace max by sup, then the core would consist of a single point, v would be in pNA, and the value φv would coincide with the single point in the core.

Under this new definition of v, the v for Example 33.11 is given by*

$$v(S) = \lambda(S) \text{ (ess. sup. } S).$$

*ess. sup. S is the essential supremum of S, i.e., the smallest number α with the property that $\lambda(S \cap [\alpha, 1]) = 0$.

The core of this v consists of a single point, namely λ. Indeed, it is easy to see that λ is in the core. Suppose that the core also contains another point, say ν. Since $\nu \neq \lambda$, there is a set S such that $\nu(S) < \lambda(S)$; let k be sufficiently large so that $1/k < \lambda(S) - \nu(S)$. Divide I into k disjoint sets each of which has essential supremum 1. For at least one of these sets--let us call it T--we must have

$$\nu(T) \leqq \nu(I)/k = 1/k.$$

Since ν is nonnegative (because $\nu(S) \geqq v(S)$), it follows that

$$\nu(S \cup T) \leqq \nu(S) + \nu(T) < \lambda(S);$$

on the other hand

$$\nu(S \cup T) \geqq v(S \cup T) = \lambda(S \cup T) \text{ ess.sup. } (S \cup T) \geqq \lambda(S) \text{ ess.sup. } T = \lambda(S).$$

This contradiction proves that the core indeed contains only the point λ

Unfortunately, though, v is not in pNA; in fact, it is not even in AC. To see this, let us define an arbitrary set function v to be <u>continuous at</u> S, where $S \in \mathcal{C}$, if for all nondecreasing sequences $\{S_i\}$ such that $\cup S_i = S$, and all nonincreasing sequences $\{S_i\}$ such that $\cap S_i = S$, we have

$$\lim v(S_i) = v(S).$$

It is easily verified that every member of AC is continuous at every $S \in \mathcal{C}$. But the v of Example 33.11 is almost never* continuous. For example, if $S = [0, \frac{1}{2}]$ and $S_i = [0, \frac{1}{2}] \cup [1 - \frac{1}{i}, 1]$, then $\{S_i\}$ is a monotone nonincreasing sequence and $\cap S_i = S$; but

$$\lim v(S_i) = \frac{1}{2} > \frac{1}{4} = v(S).$$

Therefore, $v \notin$ AC and a fortiori $v \notin$ pNA, and so Theorem G cannot be generalized to this situation.

There is, however, still some hope that Proposition 31.7 might be generalizable, i.e., that we might be able to show that the core is unique, and that its asymptotic value exists and equals the unique point in the core. In the case of Example 33.11, this is indeed the case. We have already seen that the core consists of the unique point λ. To see that the asymptotic value exists and equals λ, consider a partition of I into a large number of small sets** $S_1, \ldots, S_k$. In a random ordering of the S_i, there will with high probability be an S_i near the

*It is continuous only at those S for which ess. sup. S = 0 (i.e., $\lambda(S) = 0$) or ess. sup. S = 1.

**For definiteness one can think of intervals of equal length, but the argument goes through perfectly well without any such assumption.

beginning of the ordering whose essential supremum is close to 1. This means that with high probability most of the S_i will be contributing approximately $\lambda(S_i)$ to $v(S)$, and our assertion about the asymptotic value follows from this. Of course the argument as given here is heuristic, but the reader may convince himself that it is easily made precise.

Unfortunately, our relatively good fortune in being able to generalize Proposition 31.7 in the case of Example 33.11 does not extend to any appreciable class of games. We now bring an example of an economy satisfying all our assumptions except (31.2), in which the core of v contains many points.*

Example 33.12. Let $n = 2$, let

$$u(x, s) = \begin{cases} (x^1 + x^2) - ((x^1)^{1/s} + (x^2)^{1/s})^s & \text{when } s \in (0, 1), \\ x^1 + x^2 & \text{when } s = 0 \text{ or } s = 1, \end{cases}$$

and let

$$\underset{\sim}{a}(s) = \begin{cases} (1/2, 3/2) & \text{when } s \in [0, \frac{1}{2}], \\ (3/2, 1/2) & \text{when } s \in (\frac{1}{2}, 1]. \end{cases}$$

Here, the exact form of u is of no importance; what is needed is only that for fixed s, u be increasing, differentiable, and homogeneous of degree 1 in x, that for

*When the max in (30.1) is replaced by sup.

$s \in (0, 1)$ u be decreasing in s for fixed x, and that

$$\lim_{s \to 0} u(x, s) = \min (x^1, x^2).$$

The form of u at s = 0 and at s = 1 is of no importance.

Define a vector measure ζ by

$$\zeta(S) = \int_S \underset{\sim}{a}.$$

Then,**

$$v(S) = \begin{cases} \min(\zeta^1(S), \zeta^2(S)), & \text{when ess. inf. } S = 0 \\ u(\zeta(S), \text{ess. inf. } S), & \text{when ess. inf. } S > 0. \end{cases}$$

From this it follows that ζ^1, ζ^2, and any convex combination of ζ^1 and ζ^2 are in the core of v, and so the core contains more than one point. Therefore without (31.2) or at least some condition that guarantees that v(I) is attained, there is no hope for generalizing Proposition 31.7 either.

Proposition 31.5 cannot be extended either. Rather than describing the example in detail, we will indicate it by means of a figure.

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*ess. inf. S is the essential infimum of S; it is defined to be the largest α such that $\lambda(S \cap [0, \alpha]) = 0$

<u>Example 33.13</u>. Let $n = 2$, and let

$$f(x) = \min(x^1, x^2) + x^1 + x^2.$$

For all s, $u(x) = u(x, s)$ is defined to be $f(x)$ when x is <u>not</u> in the interior of the central region C in Fig. 7, while in the interior of C it is defined so that it is non-negative, differentiable, increasing, and $\leq f(x)$. The initial bundle $\underset{\sim}{a}$ is defined by

$$\underset{\sim}{a}(s) = \begin{cases} (1/2,\ 3/2), & \text{when } s \in [0, \tfrac{1}{2}], \\ (3/2,\ 1/2), & \text{when } s \in (\tfrac{1}{2}, 1]. \end{cases}$$

The smallest concave function that is $\geq u(x)$ is $f(x)$. Readers familiar with the methods of [A-P] (compare also Proposition 39.3) will be able to deduce without difficulty that $v = f \circ \zeta$, where ζ is given by $\zeta(S) = \int_S \underset{\sim}{a}$; this may also be seen directly via Lyapunov's theorem [L]. In any case we see that the core of v contains $2\zeta^1 + \zeta^2$, $\zeta^1 + 2\zeta^2$, and all convex combinations of these two measures.

What happens if we require (instead of (31.2)) that $u(\cdot, s)$ be concave for each fixed s? If u is not required to be of finite type, then this does not help; indeed,

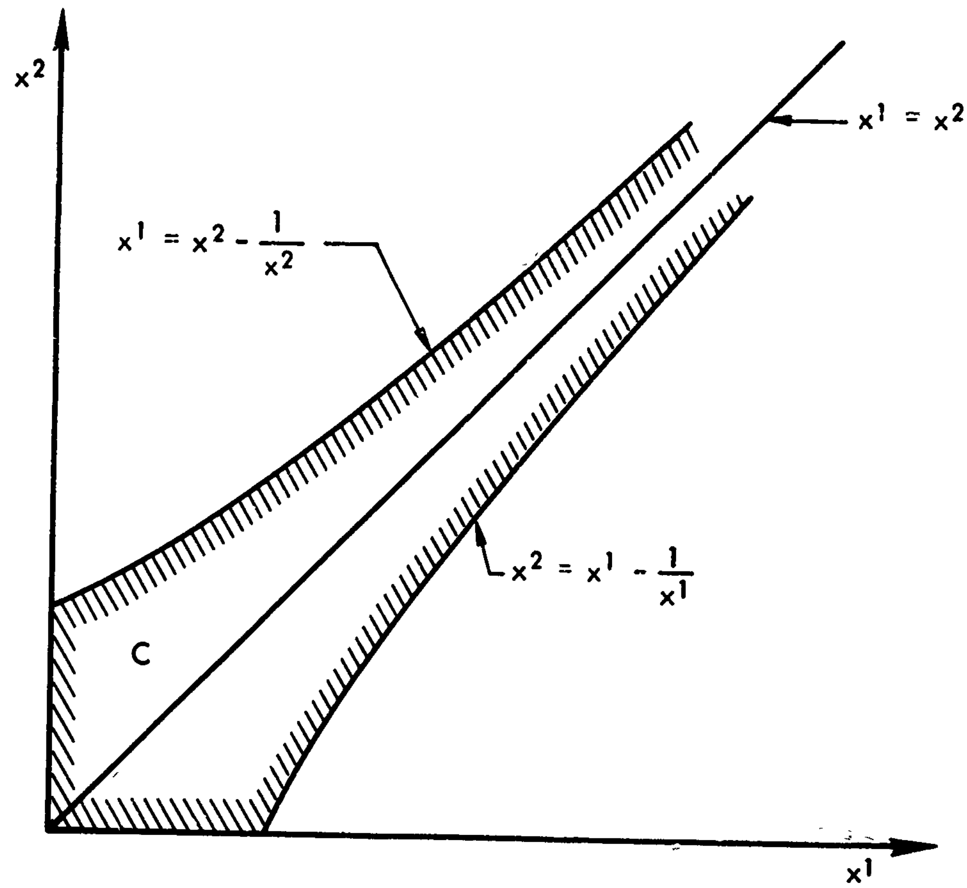

Fig.7 — The region C in Example 33.13

the u of Example 33.12 has this property.* If u is of finite type, then we are in a situation where all the v(S) are attained in spite of the fact that (31.2) does not hold. This situation will be discussed further in Section 42.

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*It is even strictly concave.

34. DISCUSSION OF THE LITERATURE

The fact that in a barter economy with a continuum of traders, the core coincides with the set of all competitive allocations has been discussed extensively in the literature; see, for example, [A_1, C, Hi, V]. The same principle, in a different form, is embodied in the theorem of [D-Sca]; there it is proved that under appropriate conditions, the core of an n-person barter economy "tends", in a certain sense, to the set of all competitive allocations as $n \to \infty$. To distinguish these two kinds of theorems, let us call the former a continuous theorem, the latter an asymptotic theorem.

Monetary economies with finitely many agents* were introduced in [Shu]. They were subsequently studied by Shapley and Shubik in a number of papers [e.g., S-S_1, S-S_2, S-S_3, S_5, S_8]. Aside from the intrinsic interest of such economies, they are interesting as special cases of the more general Walrasian barter economies;** indeed, in a number of instances, results first obtained for monetary economies were subsequently generalized to barter economies.

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*We use this term to mean either "trader" or "producer", according to the interpretation. Monetary economies with finitely many agents are defined in a manner entirely analogous to the continuous monetary economies defined in this paper.

**Monetary economies are easier to deal with than barter economies because they can be modelled as games with side payments--i.e., with a numerical characteristic function--which barter economies in general cannot. Compare the end of Section 29, also [S-S_1], p. 808.

The problem studied in this paper from a "continuous" viewpoint was studied in [S_8] from an "asymptotic" viewpoint. Specifically, consider a monetary economy with a fixed finite number m of types of agents, where unlike here, the type of an agent is determined both by his initial bundle and his utility* function. Let there be k agents of each type. Assume that the utility function of each agent is concave and differentiable (in the sense of (31.3)). Then as $k \to \infty$, the Shapley value of each trader tends, uniformly, to what he would get under an m.c.e.

This theorem can be compared and contrasted with our results in a number of respects. The most obvious difference is, of course, the fact that ours are continuous theorems, whereas that of [S_8] is an "asymptotic" one. The comparison in this case turns out to be typical of similar comparisons between continuous and asymptotic theorems in other cases.

First of all, continuous theorems are usually "cleaner" in their statement: they assert equality, whereas asymptotic theorems assert only that a certain limiting relation holds. This is exactly the situation here. To some extent, of course, the difference is illusory; in the continuous result, the limit notion is often built into the definition of the objects about which one is asserting equalities. Let us, for example, compare the main theorem of [S_8] with Proposition 31.7: in the former, one considers

*We shall use this term to mean utility function in the exchange interpretation, production function in the production interpretation.

the limit of values of a certain sequence of finite economies; in the latter, one considers immediately a continuous economy, but defines its value via a sequence of values of finite games. One may avoid such a process by using the "axiomatic" value, as in Theorem G; nevertheless, the axiomatic value appearing there does, in fact, equal the asymptotic value. The core notion is less directly related to finite games; but even this could easily be defined asymptotically. One must remember also that the whole notion of a game with a continuum of players is intuitively appealing only in the sense that it somehow approximates a large finite game; in a sense, therefore, the asymptotic approach is more direct. Thus we may sum up by saying that here, as usual in such situations, the continuous approach yields cleaner results, but is somewhat more sophisticated, conceptually as well as in its use of mathematical tools.

A more important difference, perhaps, is that asymptotic theorems usually require far stronger assumptions than continuous theorems. Outstanding among such assumptions--here as in other cases--is that the number of types of traders is a fixed finite number. Concavity (or quasi-concavity) of preferences is also often required in asymptotic theorems, but not in continuous ones, and this is the case here as well. There is one respect in which the asymptotic result of [S_8] assumes less than we do here, and that is in the behaviour of u; we assume that u is increasing

and that $u(x) = o(\|x\|)$ as $\|x\| \to \infty$, whereas neither assumption is needed in [S_8]. However if, as in [S_8], we assume concavity and finite type, then we might be able to dispense with these two assumptions* on u.

Finally, it may be remarked that the asymptotic results imply a framework within which the manner and rate of convergence can be discussed. The continuous formulation, by its nature, precludes such considerations.

*Cf. the end of Section 42.

35. THE SPACE $\mathcal{U}_1$

The proof of Theorem G will proceed as follows: First we shall prove Proposition 31.5, which assumes that u is of finite type. To go from this to Theorem G, we shall approximate to general u's by u's of finite type. Now each u may be viewed as a family of functions $u(\cdot,s)$ on Ω. We shall say that two u's are close, if roughly speaking, they are close for all but a small set of s's. But for this one needs a metric on the space of functions on Ω of which the functions $u(\cdot,s)$ are typical. In this section, we shall define such a metric, use it to define precisely the abovementioned notion of closeness between two u's, and finally, prove that any u can then be approximated by a u of finite type.

We shall assume w.l.o.g. that $\mu(I) = 1$. For $x \in \Omega$, we shall write Σx to mean $\Sigma_{i=1}^{n} x^{i}$.

Let $\mathcal{F}_0$ denote the set of all real-valued functions f on Ω that are continuous, are nondecreasing, vanish at 0, and satisfy

(35.1) $$f(x) = o(\Sigma x) \text{ as } \Sigma x \to \infty.$$

Let $\mathcal{F}_1$ denote the set of all f in $\mathcal{F}_0$ that are increasing (rather than just nondecreasing) and

(35.2) have continuous partial derivatives $f^j(x) = \partial f/\partial x^j$ at each $x \in \Omega$ for which $x^j > 0$.

Note that (35.1) is equivalent to the condition

$$f(x) = o(\|x\|) \text{ as } \|x\| \to \infty, \tag{35.3}$$

since

$$\frac{1}{n}\,\Sigma x \leqq \|x\| \leqq \Sigma x \tag{35.4}$$

for all $x \in \Omega$.

Let $\mathfrak{F}$ be $\mathfrak{F}_o$ or $\mathfrak{F}_1$. If $\mathfrak{L}$ is the linear span of $\mathfrak{F}$ (i.e., the set of all finite linear combinations of members of $\mathfrak{F}$), then we impose a norm on $\mathfrak{L}$ by

$$\|g\| = \sup_{x\in\Omega} |g(x)|/(1 + \Sigma x);$$

that this norm is finite follows from (35.1). This norm induces a metric and hence a topology on $\mathfrak{L}$, and hence on $\mathfrak{F}$.

PROPOSITION 35.5. $\mathcal{F}_1$ has a denumerable dense subset.

Proof. Let Ω' denote the one-point compactification of Ω, which is obtained from Ω by adding a point that we shall call ∞. If $f \in \mathcal{F}_1$, then the function $f(x)/(1 + \Sigma x)$ can be extended in a natural way from Ω to Ω' by defining it to be 0 at ∞; it will then be continuous on all of Ω'. Let $\mathcal{F}_1'$ be the set of all functions f' on Ω', such that $f'(\infty) = 0$ and for some $f \in \mathcal{F}_1$, we have

$$f'(x) = f(x)/(1 + \Sigma x)$$

for all $x \in \Omega$. $\mathcal{F}_1$ is in 1-1 correspondence with $\mathcal{F}_1'$ under the correspondence

$$f'(x) \leftrightarrow f(x)/(1 + \Sigma x).$$

Now $\mathcal{F}_1'$ is a subspace of the space $C(\Omega')$ of all continuous functions on Ω'; if we impose the uniform convergence metric on $C(\Omega')$, it then follows from the compactness of Ω' and the Stone-Weierstrass theorem that $C(\Omega')$ has a denumerable dense subset. But since it is metric, it

follows that it has a denumerable basis; hence also $\mathfrak{F}_1'$ has a denumerable basis, in the uniform convergence topology. Now the above 1-1 correspondence takes the uniform convergence topology on $\mathfrak{F}_1'$ onto the norm topology on $\mathfrak{F}_1$ that we have defined previously; hence, $\mathfrak{F}$ has a denumerable basis in that topology, and so also a denumerable dense subset. This completes the proof of Proposition 35.5.

Let $\mathfrak{F}$ be $\mathfrak{F}_o$ or $\mathfrak{F}_1$, $\mathfrak{s}$ its linear span. We remark that from the compactness of Ω' and from the fact that $g(x) = o(\Sigma x)$ for all $g \in \mathfrak{s}$, it follows that the sup in the definition of $\|g\|$ is attained, so that we may write

$$\|g\| = \max_{x\in\Omega} |g(x)|/(1 + \Sigma x).$$

Let $\mathcal{U}_o$ be the space of all functions u on $\Omega \times I$ that satisfy (31.1) and (31.2), and such that $u(\cdot, s)$ is non-decreasing and continuous on Ω and vanishes at 0. Let $\mathcal{U}_1$ be the space of all $u \in \mathcal{U}_o$ satisfying (31.3), and such that $u(\cdot, s)$ is increasing for each fixed s. Note that if $u \in \mathcal{U}_i$ then $u(\cdot, s) \in \mathfrak{F}_i$, where $i = 0$ or 1.

Let $u \in \mathcal{U}_o$. For $s \in I$, we write u_s for the function on Ω whose value at x is $u(x, s)$. For $\delta > 0$, a δ-<u>approximation</u> to u is defined to be a member $\hat{u}$ of $\mathcal{U}_o$ such that

$\|\hat{u}_s - u_s\| \leqq \delta$ for all s except possibly a set of μ-measure at most δ, in which

$$\hat{u}_s(x) = \sqrt{\Sigma x}.$$

PROPOSITION 35.6. *For every* $\delta > 0$ *and* $u \in \mathcal{U}_1$, *there is a* $\hat{u} \in \mathcal{U}_1$ *that is a* δ-*approximation to* u *and is of finite type*.*

Proof. Let $\{f_1, f_2, \ldots\}$ be a denumerable dense subset of $\mathfrak{F}_1$ (Proposition 35.5). Let $\delta > 0$ be given. For each s in I, $u_s \in \mathfrak{F}_1$; let $\underset{\sim}{i}(s)$ be the first i such that

$$\|u_s - f_{\underset{\sim}{i}(s)}\| \leqq \delta.$$

It may be seen that $\underset{\sim}{i}$ is Borel measurable. For each k, define $u^k \in \mathcal{U}_1$ by

$$u_s^k = \begin{cases} f_{\underset{\sim}{i}(s)}(x), & \text{if } \underset{\sim}{i}(s) \leqq k, \\ \sqrt{\Sigma x}, & \text{otherwise.} \end{cases}$$

*See Section 31.

Let

$$S_i = \{s \in I : \underset{\sim}{i}(s) = i\}.$$

Clearly, $\cup_{i=1}^{\infty} S_i = I$, and hence for k sufficiently large,

$$\mu(I \setminus \cup_{i=1}^{k} S_i) \leq \delta;$$

for such k, u^k is a δ-approximation to u of finite type. This completes the proof of Proposition 35.6.

36. FURTHER PREPARATIONS

In this section we shall introduce some notation and quote some results from [A-P]* that will be used throughout the sequel. The most important of these results give sufficient conditions for the attainability of the max in expressions of the form (30.1), and necessary and sufficient conditions for a specific measurable $\underset{\sim}{x}$ actually to attain this max.

Let u be a Borel-measurable function on $\Omega \times I$. If $\underset{\sim}{x}$ is a μ-integrable function from I to Ω, we will abuse our notation by writing $u(\underset{\sim}{x})$ for the function on I whose value at s is $u(\underset{\sim}{x}(s), s)$. For all $a \in \Omega$ and $S \in \mathcal{C}$, write

$$u_S(a) = \max \{\textstyle\int_S u(\underset{\sim}{x}) : \int_S \underset{\sim}{x} = a,\ \underset{\sim}{x}(s) \in \Omega \text{ for all } s\}.$$

We shall say that $u_S(a)$ <u>is attained at</u> $\underset{\sim}{x}$ if $\underset{\sim}{x}$ is an integrable function from I to Ω such that $\int_S \underset{\sim}{x} = a$ and $\int_S u(\underset{\sim}{x}) = u_S(a)$.

<u>PROPOSITION 36.1</u>. <u>Let</u> $u \in \mathcal{U}_0$. <u>Then for all</u> S <u>and</u> a, $u_S(a)$ <u>exists, i.e., the max is attained and is finite</u>.

*In citing these results, the more general hypothesis of upper-semicontinuity in [A-P] has been replaced by the present assumption of continuity.

This is essentially the main theorem of [A-P]. Note that

$$v(S) = u_S(\int_S \underset{\sim}{a}).$$

Next, we explain the concept of concavification. Let f be a nonnegative real-valued function on Ω and let

$$F = \{(\nu, x) \in E^1 \times \Omega : 0 \leqq \nu \leqq f(x)\}.$$

Let F^* be the convex hull of F. If there is a function f^* on Ω such that

$$F^* = \{(\nu, x) \in E^1 \times \Omega : 0 \leqq \nu \leqq f^*(x)\},$$

then f is said to be _spannable_, and f^* is called the _concavication_* of f; clearly f^* is unique and concave. If f is concave, then f is spannable and $f^* = f$.

PROPOSITION 36.2. _If_ $f \in \mathfrak{F}_o$, _then_ f _is spannable and_ f^* _is nondecreasing and continuous_.

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*In this paper the word "concavification" applies only to spannable functions. This coincides with the usage of [A-P]; it differs slightly from that of [S-S_1] where concavification is defined in terms of the _closure_ of the convex hull.

This is essentially Proposition 3.1 of [A–P].

If $u \in \mathcal{U}_o$, then $u_s \in \mathcal{F}_o$ for all s, and so u^*_s is defined. Define a function u* on $\Omega \times I$ by

$$u^*(x, s) = u_s{}^*(x).$$

Then $u^*{}_s = u_s{}^*$, and so we will henceforth write u^*_s for their joint value.

PROPOSITION 36.3. Let $u \in \mathcal{U}_o$. Then $u^* \in \mathcal{U}_o$, and for all S and a,

$$u^*{}_S(a) = u_S(a).$$

In particular, it follows that u_S is concave on Ω.

This is an immediate consequence of [A–P], specifically of Lemma 3.3 and 3.5 and Propositions 3.1 and 4.1 of that paper. From the concavity of u_S and $u^*{}_S = u_S$ if follows that $u^*{}_S = u_S{}^*$; so we will henceforth write u^*_S for their joint value.

PROPOSITION 36.4. *Let* $u \in \mathcal{U}_0$, *let* $a \in \Omega$ *be* $\gg 0$, *let* $S \in \mathcal{C}$ *and let* $\underset{\sim}{x}$ *be a measurable function from* I *to* Ω. *Then a necessary and sufficient condition for* $u_S(a)$ *to be attained at* $\underset{\sim}{x}$, *i.e., for*

$$\int_S u(\underset{\sim}{x}) = u_S(a) \text{ and } \int_S \underset{\sim}{x} = a,$$

is that there be a $p \in \Omega$ *such that*

$$u(x, s) - u(\underset{\sim}{x}(s), s) \leqq p \cdot (x - \underset{\sim}{x}(s))$$

for all $x \in \Omega$ *and almost all* $s \in S$. *If* u *is increasing for each fixed* s, *then* $p \in \Omega$ *(i.e.,* $p \geqq 0$*) may be replaced by* $p > 0$. *If* $u \in \mathcal{U}_1$, *then for* $i = 1, \ldots, n$,

$$p^i = [\partial u / \partial x^i]_{x = \underset{\sim}{x}(s)}$$

for almost all s *such that* $\underset{\sim}{x}^i(s) > 0$.

This is essentially Proposition 5.1 of [A-P].

We close this section with a statement of the Measurable Choice Theorem.

PROPOSITION 36.5. *Let* $(X, \mathcal{X})$ *be a standard measurable space, i.e., one that is isomorphic to* $([0, 1], \mathcal{B})$. *Let* $\mathcal{S}$ *be a subset of* $I \times X$ *that is measurable in the product* σ-*field* $\mathcal{C} \times \mathcal{X}$, *and whose projection on* I *is* I. *Then there is a measurable function* $g : I \to X$ *such that for almost all* s, $(g(s), s) \in \mathcal{S}$.

This theorem is due to von Neumann [VN, p. 448, Lemma 5]. Von Neumann's proof uses Assumption 2.1, namely that $(I, \mathcal{C})$ is also standard, but the theorem remains true without this assumption; see [A_6].

37. BASIC PROPERTIES OF δ-APPROXIMATIONS

The main goal of this section is Lemma 37.8, in which it is shown that for given u, there is a fixed integrable function η that bounds all coordinates of all functions $\underset{\sim}{y}$ that maximize $\hat{u}_S(b)$, whenever S is not too small, b is bounded away from 0 and ∞, and $\hat{u}$ is a sufficiently good δ-approximation to u. The existence of such a fixed η is important in, among other things, compactness arguments in many places in the sequel. One example of such a use is in Proposition 37.13, in which the continuity of u_S on Ω is established (the difficulty, of course, occurs on the boundary of Ω).

<u>LEMMA 37.1</u>. <u>Let</u> $u \in \mathcal{U}_0$. <u>For each</u> $\delta > 0$ <u>let</u> η_δ <u>be an integrable function with</u> $\eta_\delta(s) \geq 1$ <u>for each</u> s, <u>such that</u>

$$u(x, s) < \delta \Sigma x$$

<u>and</u>

$$\sqrt{\Sigma x} < \delta \Sigma x$$

<u>whenever</u> $\Sigma x \geq \eta_\delta(s)$. <u>Then if</u> $\hat{u}$ <u>is a</u> δ-<u>approximation to</u> u, <u>then</u>

$$\hat{u}(x, s) < 3\delta \Sigma x$$

<u>whenever</u> $\Sigma x \geq \eta_\delta(s)$.

<u>Proof</u>. We have $\hat{u}(x, s) = \sqrt{\Sigma x}$ or

$$|\hat{u}(x, s) - u(x, s)| \leq \delta(1 + \Sigma x).$$

In the first case there is nothing to prove, and in the second case, if $\Sigma x \geqq \eta_\delta(s)$, then by using $\eta_\delta(s) \geqq 1$ we obtain

$$\hat{u}(x, s) \leqq \delta(1 + \Sigma x) + u(x, s) < 2\delta\Sigma x + \delta\Sigma x = 3\delta\Sigma x.$$

This completes the proof of Lemma 37.1.

For each $f \in \mathfrak{F}_0$ and $x \in \Omega$, let

$$P(x; f) = \{p \in \Omega : f(y) - f(x) \leqq p\cdot(y - x) \text{ for all } y \in \Omega\}.$$

Let $\xi(x; f)$ be the infimum* value that any coordinate of any point in $P(x; f)$ can achieve; more precisely,

$$\xi(x; f) = \min_i \inf \{p^i : p \in P(x; f)\}.$$

If x is an interior point of Ω and f is concave and differentiable, then $P = P(x; f)$ contains precisely one point, namely the gradient $f'(x)$; in that case $\xi = \xi(x; f)$ is simply the smallest partial derivative $f^i(x)$. If f is

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*As usual, the infimum of the empty set is taken to be $+\infty$; thus if $P(x; f) = \emptyset$ then $\xi(x; f) = +\infty$.

differentiable but not concave, then P can contain at most a single point (namely the gradient), but may also be empty. If it is concave but not necessarily differentiable, then P is non-empty, and consists of the normal vectors to hyperplanes that support the subgraph* of f at the point (x, f(x)). This, in fact, is the general characterization of P, also when f need be neither differentiable nor concave, and when x may be on the boundary of Ω.

LEMMA 37.2. *Let* $u \in \mathcal{U}_0$ *be increasing for each fixed* s. *Then for each* $\epsilon > 0$ *and each real* α *there is a* $\delta > 0$ *such that if* $\hat{u} \in \mathcal{U}_0$ *is a* δ*-approximation to* u, $S \in \mathcal{C}$ *is such that* $\mu(S) \geq \epsilon$, *and* $\underset{\sim}{x}$ *is an integrable function from* I *to* Ω *such that*

$$\varsigma(\underset{\sim}{x}(s);\ \hat{u}_s) < \delta$$

for all $s \in S$, *then*

$$\Sigma\int_S \underset{\sim}{x} > \alpha.$$

*The set of all points underneath or on the graph.

Proof. First we prove:

(37.3) If C is a compact subset of Ω, then for each s there is a $\delta = \delta(C, s) > 0$ such that $\xi(x; f) \geqq \delta$ for all $x \in C$ and all $f \in \mathfrak{F}_0$ such that $\|f - u_s\| \leqq \delta$.

Indeed, if not, let $\{x_1, x_2, \ldots\}$ be a sequence in C, and $\{f_1, f_2, \ldots\}$ a sequence in $\mathfrak{F}_1$ such that $\|f_k - u_s\| \to 0$ and $\xi(x_k; f_k) \to 0$. Let x_0 be a limit point of $\{x_k\}$, w.l.o.g. a limit. Further, assume w.l.o.g. that $\xi(x_k; f_k)$ is "assumed at p^1" for all j, i.e. that

$$\inf\{p^1 : p \in P(x_k; f_k)\} = \xi(x_k; f_k).$$

It follows that for $k = 1, 2, \ldots$, there is a $p_k \in P(x_k; f_k)$ such that $p_k^1 < \xi(x_k; f_k) + \frac{1}{k}$; then $p_k^1 \to 0$, and

$$f_k(y) - f_k(x_k) \leqq p_k \cdot (y - x_k)$$

for all $y \in \Omega$. Now for $k = 0, 1, 2, \ldots$ set $y_k = x_k + (1, 0, \ldots, 0)$. Since $y_k \in \Omega$, we have

$$(37.4) \qquad f_k(y_k) - f_k(x_k) \leqq p_k \cdot (y_k - x_k) = p_k^1 \to 0.$$

Now

$$u(y_k, s) - f_k(y_k) \leqq (1 + \Sigma y_k)\|f_k - u_s\|$$

and

$$f_k(x_k) - u(x_k, s) \leqq (1 + \Sigma x_k)\|f_k - u_s\|.$$

Hence

$$u(y_k, s) - u(x_k, s) \leqq [f_k(y_k) - f_k(x_k)]$$

$$+ [(1 + \Sigma y_k) + (1 + \Sigma x_k)]\|f_k - u_s\|.$$

Since C is bounded and $x_k \in C$, it follows that $1 + \Sigma x_k$ is bounded; hence also $1 + \Sigma y_k = 2 + \Sigma x_k$ is bounded. Since $\|f_k - u_s\| \to 0$ by assumption, the second term of the right side of this inequality approaches 0. The first term is nonnegative because f is nondecreasing, and so by (37.4), it approaches 0 as well. Hence the left side tends to 0,

and from the continuity of u it follows that $u(y_0, s) - u(x_0, s) \leq 0$, contradicting the fact that u_s is increasing. This contradiction establishes (37.3).

Let us now set

$$(37.5) \quad \gamma = \gamma(s,\delta) = \inf\{\Sigma x : (\exists f \in \mathfrak{F}_0)(\|f-u_s\| \leq \delta \text{ and } \xi(x;f) < \delta)\}.$$

Clearly γ is non-decreasing as δ decreases. Suppose γ is bounded as $\delta \to 0$, say $\gamma < \gamma_0$ ($= \gamma_0(s)$). Then in the compact set

$$C = \{x \in \Omega : \Sigma x \leqq \gamma_0\},$$

$\xi(x; f)$ comes arbitrarily close to 0 for f arbitrarily close in norm to u_s, contradicting (37.3). Hence for each s,

$$(37.6) \qquad \gamma(s, \delta) \to \infty \quad \text{as} \quad \delta \to 0.$$

If the lemma is false, then for each k there is a set S_k of measure $\geq \epsilon$, a $\frac{1}{k}$-approximation $\hat{u}$ to u, and an integrable function $\underset{\sim}{x}_k$ such that

$$\xi(\underset{\sim}{x}_k(s);\ \hat{u}_s) < \frac{1}{k}$$

for $s \in S_k$ and

$$\Sigma\int_{S_k} \underset{\sim}{x}_k \leq \alpha. \tag{37.7}$$

Then $\|\hat{u}_s - u_s\| \leq 1/k$ for all s except for s in a set V_k, where $\mu(V_k) \leq 1/k$. From (37.5) it then follows that for $s \in S_k \backslash V_k$, we have $\Sigma\underset{\sim}{x}_k(s) \geq \gamma(s, 1/k)$, and so from (37.6) we deduce that for such s,

$$\Sigma\underset{\sim}{x}_k(s) \to \infty \quad \text{as} \quad k \to \infty.$$

Now define $\underset{\sim}{g}_k : I \to E^1$ by

$$\underset{\sim}{g}_k(s) = \begin{cases} \Sigma\underset{\sim}{x}_k(s) & \text{if } s \in S_k \backslash V_k \\ k & \text{otherwise.} \end{cases}$$

Then for each $s \in I$, $g_k(s) \to \infty$ as $k \to \infty$. Hence from Egoroff's theorem it follows that $g_k(s) \to \infty$ as $k \to \infty$ uniformly for s in a subset U of I of measure $1 - \frac{1}{2}\epsilon$; thus for $s \in U$ we have $g_k(s) \geqq \gamma_k \to \infty$, say. In particular, it follows that for $s \in (S_k \cap U)\backslash V_k$, we have

$$\Sigma \underset{\sim}{x}_k(s) \geqq \gamma_k \to \infty.$$

From this and (37.7) we set

$$\alpha \geqq \Sigma \int_{S_k} \underset{\sim}{x}_k = \int_{S_k} \Sigma \underset{\sim}{x}_k \geqq \int_{(S_k \cap U)\backslash V_k} \Sigma \underset{\sim}{x}_k \geqq \left(\frac{\epsilon}{2} - \frac{1}{k}\right)\gamma_k \to \infty,$$

an absurdity. This completes the proof of Lemma 37.2.

LEMMA 37.8. Let $u \in \mathcal{U}_0$ be increasing for each fixed s. Then for each $\epsilon > 0$ and each $\alpha \geqq 0$ there is a $\delta > 0$ and an integrable function $\underset{\sim}{\eta}$ such that if $S \in \mathcal{C}$ is such that $\mu(S) \geqq \epsilon$, b in Ω satisfies $\Sigma b < \alpha$, $\hat{u} \in \mathcal{U}_0$ is a δ-approximation to u, and $\hat{u}_S(b)$ is attained at $\hat{\underset{\sim}{y}}$, then

$$\hat{\underset{\sim}{y}}(s) \leqq \underset{\sim}{\eta}(s)e$$

for almost all $s \in S$.

Proof. Lemma 37.2 yields a δ—which we call δ_1 to distinguish it from the δ of this lemma—that obeys the conclusions of that lemma. Let $\delta = \delta_1/3n$. Because of Lemma 37.1, there is an integrable function $\underset{\sim}{\eta}$ such that $\hat{u}(z, s) < 3\delta\Sigma z$ whenever $\Sigma z \geqq \underset{\sim}{\eta}(s)$ and $\hat{u}$ is a δ-approximation to u. We will prove that this δ and $\underset{\sim}{\eta}$ satisfy Lemma 37.8.

Suppose that they do not. Then there is an S with $\mu(S) \geqq \epsilon$, a δ-approximation $\hat{u}$ to u, a j with $1 \leqq j \leqq n$, a b in Ω with $\Sigma b < \alpha$, and a subset U of S of positive measure such that

$$\hat{\underset{\sim}{y}}^j(s) > \underset{\sim}{\eta}(s)$$

for all $s \in U$. W.l.o.g. we may assume that $\hat{\underset{\sim}{y}}^j(s) = \max_i \hat{\underset{\sim}{y}}^i(s)$ for all s in U. Now for fixed but arbitrary s_0 in U, let x be the vector whose j^{th} coordinate vanishes and all of whose other coordinates are equal to the corresponding coordinates of $\hat{\underset{\sim}{y}}(s_0)$. Let p be the price vector corresponding to $\hat{\underset{\sim}{y}}$ and S in accordance with Proposition 36.4. Then

$$-\hat{u}(\hat{\underset{\sim}{y}}(s_0), s_0) \leqq \hat{u}(x, s_0) - \hat{u}(\hat{\underset{\sim}{y}}(s_0), s_0) \leqq p\cdot(x - \hat{\underset{\sim}{y}}(s_0))$$

$$= p^j(x^j - \hat{\underset{\sim}{y}}^j(s_0)) = -p^j\hat{\underset{\sim}{y}}^j(s_0).$$

But since $\Sigma\hat{y}(s_0) \geq \hat{y}^j(s_0) > \eta(s_0)$, it follows that

$\hat{u}(y(s_0), s_0) < 3\delta \, \Sigma \, \hat{y}^i(s_0) \leq 3n\delta \, \max_i \, \hat{y}^i(s_0) = \delta_1 \hat{y}^j(s_0)$;

hence $p^j \hat{y}^j(s_0) < \delta_1 \hat{y}^j(s_0)$, and therefore $p^j < \delta_1$. But we have chosen p so that $p \in P(y(s); \hat{u}_s)$ for all $s \in S$. Hence for $s \in S$, we have

$$\varsigma(\hat{y}(s); \hat{u}_s) \leq p^j < \delta_1.$$

Furthermore, since $\delta = \delta_1/3n < \delta_1$ and $\hat{u}$ is a δ–approximation to u, it is a fortiori a δ_1–approximation. Since $\mu(S) \geq \epsilon$, it follows from Lemma 37.2 that $\Sigma\int_S \hat{y} > \alpha > \Sigma b$, contradicting $\int_S \hat{y} = b$. This proves Lemma 37.8.

<u>LEMMA 37.9</u>. <u>Let</u> $u \in \mathcal{U}_0$. <u>Then for any</u> ϵ, <u>there is a</u> μ–<u>integrable real function</u> ζ, <u>such that for all</u> s <u>in</u> I <u>and all</u> x <u>in</u> Ω,

$$u(x, s) \leq \epsilon(\zeta(s) + \Sigma x).$$

<u>Proof</u>. Let ζ be an integrable real function such that $u(y, s) \leq \epsilon\Sigma y$ whenever $\Sigma y \geq \zeta(s)$; such a ζ exists because

(35.1) must hold integrably in s. Then because u_s is non-decreasing, we have

$$u(x, s) \leqq u(x + \underset{\sim}{\zeta}\frac{e}{n}, s) \leqq \epsilon(\underset{\sim}{\zeta}(s) + \Sigma x),$$

as was to be proved.

COROLLARY 37.10. Let $u \in \mathcal{U}_0$. Then if $\underset{\sim}{x}$ is integrable, so is $u(\underset{\sim}{x})$.

PROPOSITION 37.11. Let $u \in \mathcal{U}_0$. Then for each $\epsilon > 0$, there is a $\delta > 0$, such that if $\hat{u} \in \mathcal{U}_0$ is a δ-approximation to u, then for all $S \in \mathcal{C}$ and all $b \in \Omega$ we have

$$|u_S(b) - \hat{u}_S(b)| < \epsilon(1 + \Sigma b).$$

Proof. Let $u_1(x, s) = \sqrt{\Sigma x}$. Apply Lemma 37.9 using $\frac{\epsilon}{3}$ instead of ϵ, both to u and to u_1, obtaining functions $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta}_1$ with

$$u(x, s) \leqq \frac{\epsilon}{3}(\underset{\sim}{\zeta}(s) + \Sigma x)$$

$$u_1(x, s) \leqq \frac{\epsilon}{3}(\underset{\sim}{\zeta}_1(s) + \Sigma x).$$

Next, choose δ sufficiently small so that $\int_U(\underset{\sim}{\zeta} + \underset{\sim}{\zeta}_1) \leqq 1$ whenever $\mu(U) \leqq \delta$, and also so that $\delta \leqq \frac{\epsilon}{3}$. Letting U be the exceptional set in the definition of δ-approximation, we obtain for any $\underset{\sim}{x}$,

$$\int_S |u(\underset{\sim}{x}) - \hat{u}(\underset{\sim}{x})| \leqq \int_{S\setminus U} \frac{\epsilon}{3}(1 + \Sigma \underset{\sim}{x}) + \int_{U\cap S} (u(\underset{\sim}{x}) + u_1(\underset{\sim}{x}))$$

$$\leqq \frac{\epsilon}{3}\int(1 + \Sigma\underset{\sim}{x}) + \int_U \frac{\epsilon}{3}(\underset{\sim}{\zeta} + \underset{\sim}{\zeta}_1) + \int_U \frac{2\epsilon}{3}\Sigma\underset{\sim}{x}$$

$$\leqq \frac{\epsilon}{3}(1 + \Sigma\int\underset{\sim}{x}) + \frac{2\epsilon}{3}(1 + \Sigma\int\underset{\sim}{x}) = \epsilon(1 + \Sigma\int\underset{\sim}{x}).$$

Now let $u_S(b)$ and $\hat{u}_S(b)$ be achieved at $\underset{\sim}{y}$ and $\hat{\underset{\sim}{y}}$ respectively. Then $b = \int_S\underset{\sim}{y} = \int_S\hat{\underset{\sim}{y}}$, and we have

$$u_S(b) = \int_S u(\underset{\sim}{y}) \geqq \int_S u(\hat{\underset{\sim}{y}}) = \int_S \hat{u}(\hat{\underset{\sim}{y}}) + \int_S (u(\hat{\underset{\sim}{y}}) - \hat{u}(\hat{\underset{\sim}{y}}))$$

$$\geqq \int_S \hat{u}(\hat{\underset{\sim}{y}}) - \int_S |u(\hat{\underset{\sim}{y}}) - \hat{u}(\hat{\underset{\sim}{y}})| \geqq \int_S \hat{u}(\hat{\underset{\sim}{y}}) - \epsilon(1 + \Sigma\int_S\hat{\underset{\sim}{y}})$$

$$= \hat{u}_S(b) - \epsilon(1 + \Sigma b).$$

Hence $\hat{u}_S(b) - u_S(b) \leqq \epsilon(1 + \Sigma b)$. Similarly

$$\hat{u}_S(b) = \int_S \hat{u}(\hat{y}) \geqq \int_S \hat{u}(y) = \int_S u(y) + \int_S (\hat{u}(y) - u(y))$$

$$\geqq \int_S u(y) - \epsilon(1 + \Sigma\int_S y) = u_S(b) - \epsilon(1 + \Sigma b),$$

and so

$$u_S(b) - \hat{u}_S(b) \leqq \epsilon(1 + \Sigma b).$$

This completes the proof of Proposition 37.11.

We close this section with a proposition (Proposition 37.13) which, though not directly connected with the concept of δ-approximation, is a consequence of Lemma 37.8. First we require another lemma.

<u>LEMMA 37.12</u>. <u>Let</u> $f \in \mathfrak{F}_0$ <u>be increasing</u>. <u>Then the concavification</u> f^* <u>of</u> f <u>is also increasing</u>.

<u>Proof</u>. Let $x \in \Omega$. Since f is spannable (Proposition 36.2), there exist points $x_1, \ldots, x_k$ in Ω, and positive numbers $\alpha_1, \ldots, \alpha_k$ summing to 1, such that

$$\Sigma_{i=1}^k \alpha_i x_i = x$$

and

$$\Sigma_{i=1}^{k} \alpha_i f(x_i) = f^*(x).$$

If $y \geq x$, then there is a $z \geq 0$ such that $y = x + z$. We then have

$$\Sigma_{i=1}^{k} \alpha_i (x_i + z) = x + z = y$$

and so by the concavity of f^* and the fact that f is increasing, we get

$$f^*(y) \geqq \Sigma_{i=1}^{k} \alpha_i f^*(x_i + z) \geqq \Sigma_{i=1}^{k} \alpha_i f(x_i + z) > \Sigma \alpha_i f(x_i)$$
$$= f^*(x).$$

This completes the proof of Lemma 37.12.

<u>PROPOSITION 37.13</u>. <u>Let</u> $u \in \mathcal{U}_0$ <u>be increasing for each</u> s. <u>Then for each</u> $S \in \mathcal{C}$, u_S <u>is continuous on</u> Ω.

<u>Proof</u>. Let $b \in \Omega$; we wish to prove that u_S is continuous at b. Let

$$L = \{i : b^i > 0\}, \quad M = \{i : b^i = 0\},$$

and let

$$\Omega^L = \{a \in \Omega : a^i = 0 \quad \text{for all } i \in M\},$$
$$\Omega^M = \{a \in \Omega : a^i = 0 \quad \text{for all } i \in L\};$$

thus $b \in \Omega^L$. Our proof will proceed in two stages: First, we show that

(37.14) $\quad u_S | \Omega^L$ is continuous at b.

Second, setting

$$\Omega_b^L = \{a \in \Omega^L : b/2 \leqq a \leqq 2b\},$$

we shall show that

(37.15) $\quad$ for every ϵ there is a δ such that if $C \in \Omega^M$, $\|C\| \leqq \delta$, and $a \in \Omega_b^L$, then $u_S(a + c) - u_S(a) < \epsilon$.

Together, (37.14) and (37.15) prove the desired continuity of u_S at b.

In proving (37.14), we will never "leave" the space Ω^L; therefore we may assume w.l.o.g. that $\Omega^L = \Omega$, i.e. that $b > 0$. Then (37.14) turns into the assertion that u_S is continuous at b. By Proposition 36.3, u_S is concave on Ω, and since $b > 0$, it follows that b is in the interior of Ω. Since every concave function is continuous in the interior of its domain of definition, it follows that u_S is continuous at b, and so (37.14) is proved.

Next, we prove (37.15). By Proposition 36.3 and Lemma 37.12, we may assume w.l.o.g. that u is concave for each fixed s (otherwise, replace it by its concavification u*). Suppose now that (37.15) is false. Then we can find an $\epsilon > 0$, a sequence $\delta_j \to 0$, and sequences $\{c_j\}$ and $\{a_j\}$ such that $\|c_j\| \leq \delta_j$, $a_j \in \Omega^L_\alpha$, and

$$(37.16) \qquad u_S(a_j + c_j) - u_S(a_j) \geq \epsilon.$$

Since a_j is in Ω^L_b, which is compact, it follows that $\{a_j\}$ has a limit point a in Ω^L_b; w.l.o.g. let it be the limit. Note that since $a \in \Omega^L_b$, we have $a^i > 0$ for all $i \in L$; hence applying (37.14) to a instead of b, we get that

$$u_S(a_j) \to u_S(a) \tag{37.17}$$

as $j \to \infty$.

Now let $u_S(a_j + c_j)$ be attained at $\underset{\sim}{y}_j$. From Lemma 37.8 it follows that there is an integrable function $\underset{\sim}{\eta}$ such that $\underset{\sim}{y}_j(s) \leqq \underset{\sim}{\eta}(s)e$ for all j and almost all s. The space of all integrable functions on S can be considered as $L^1(S \times \{1, \ldots, n\})$. Since the set of all $\underset{\sim}{x}$ in this space such that $0 \leqq \underset{\sim}{x}(s) \leqq \underset{\sim}{\eta}(s)e$ a.e. is weakly sequentially compact*, it follows that the sequence $\{\underset{\sim}{y}_n\}$ has a subsequence that is weakly convergent, say to $\underset{\sim}{y}$. Then there is a sequence of functions converging strongly (i.e., in the L^1-norm) to $\underset{\sim}{y}$, each one of which is a (finite) convex combination of $\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots$ [Dun-S, p. 422, Corollary V.3.14]. Now every strongly convergent sequence in L^1 has a subsequence that converges a.e. to the same limit; so there is a sequence $\{z_j\}$ of convex combinations of $\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots$ that converges a.e. to $\underset{\sim}{y}$. Since $\underset{\sim}{y}_j(s) \leqq \underset{\sim}{\eta}(s)e$ a.e., it follows also that $\underset{\sim}{z}_j(s) \leqq \underset{\sim}{\eta}(s)e$ a.e. Hence $u(\underset{\sim}{z}_j)$ is pointwise $\leqq u(\underset{\sim}{\eta}e)$, which is integrable by Corollary 37.10. Moreover, from the continuity of $u(\cdot, s)$ for each fixed s,

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*[Dun-S], p. 292, Theorem IV.8.9.

it follows that $u(\underset{\sim}{z}_j(s), s) \to u(\underset{\sim}{y}(s), s)$ a.e. as $j \to \infty$. Hence the Lebesgue dominated convergence theorem applies, and we deduce

$$(37.18) \qquad \int_S u(\underset{\sim}{z}_j) \to \int_S u(\underset{\sim}{y}).$$

Now from the concavity of u for each fixed s and the fact the the $\underset{\sim}{z}_j$ are convex combinations of the $\underset{\sim}{y}_j$, it follows that

$$\int_S u(\underset{\sim}{z}_j) \geq \min_j \int_S u(\underset{\sim}{y}_j) = \min_j u_S(a_j + c_j).$$

Hence by (37.18), it follows that

$$\int_S u(\underset{\sim}{y}) \geq \min_j u_S(a_j + c_j).$$

But if we had chopped off any finite number of terms from the originally given sequences $\{a_j\}$ and $\{c_j\}$, this would not have changed $\underset{\sim}{y}$ nor any of the foregoing considerations. Hence for all k we have

$$\int_S u(\underset{\sim}{y}) \geq \min_{j \geq k} u_S(a_j + c_j),$$

and letting $k \to \infty$, we deduce

$$\int_S u(\underset{\sim}{y}) \geqq \liminf_{j\to\infty} u_S(a_j + c_j).$$

Applying (37.16) and (37.17), we then deduce

$$(37.19) \qquad \int_S u(\underset{\sim}{y}) \geqq u_S(a) + \epsilon.$$

On the other hand, since $\underset{\sim}{y}_j \to \underset{\sim}{y}$ weakly, we have

$$\int_S \underset{\sim}{y} = \lim_{j\to\infty} \int_S \underset{\sim}{y}_j = \lim_{j\to\infty} (a_j + c_j) = a + \lim_{j\to\infty} c_j = a.$$

Thus by definition we must have $\int_S u(\underset{\sim}{y}) \leqq u_S(a)$, in contradiction to (37.19). This completes the proof of Proposition 37.13.

38. THE DERIVATIVES OF THE FUNCTION u_S

In this section we shall establish the existence and some continuity properties of the derivatives of the function u_S.

PROPOSITION 38.1. *Let* $u \in \mathcal{U}_1$ *and* $S \in \mathcal{C}$. *Then for each* j *such that* $1 \leq j \leq n$, *the partial derivative* $u_S^j = \partial u_S/\partial x^j$ *exists at each point* $b \in \Omega$ *such that* $b^j > 0$.

Proof. Without loss of generality let $j = 1$. Because of the concavity of u_S (Proposition 36.3),

$$\alpha = \lim_{\delta \to 0+}(u_S(b + \delta e_1) - u_S(b))/\delta$$

and

$$\beta = \lim_{\delta \to 0-}(u_S(b + \delta e_1) - u_S(b))/\delta$$

both exist, though they may a priori be different; in any case we have $\alpha \leq \beta$. If $\alpha = \beta$ our theorem is proved, so let us assume $\alpha < \beta$.

We now show that*

$$(38.2) \qquad u_S(b + \gamma e_1) - u_S(b) \leq \min(\alpha\gamma, \beta\gamma)$$

*The right side of (38.2) is $\alpha\gamma$ when $\gamma > 0$ and $\beta\gamma$ when $\gamma < 0$.

for all $\gamma > -\beta^1$. Indeed, suppose that

$$u_S(b + \gamma e_1) - u_S(b) > \alpha\gamma$$

for some $\gamma > 0$ or

$$u_S(b + \gamma e_1) - u_S(b) > \beta\gamma$$

for some $\gamma < 0$. In the first case we will have

$$u_S(b + \gamma e_1) - u_S(b) = \alpha'\gamma \tag{38.3}$$

for some $\alpha' > \alpha$ and some $\gamma > 0$. Now the left side of (38.3) is a concave function of γ that vanishes for $\gamma = 0$, and hence

$$u_S(b + \delta e_1) - u_S(b) \geqq \alpha'\delta$$

for all δ such that $0 \leqq \delta \leqq \gamma$. Hence the right hand partial derivative of u_S at b is $\geqq \alpha' > \alpha$, contradicting the fact that it equals α. In the second case a contradiction is similarly obtained.

Suppose now that $u_S(b)$ is attained at $\mathbf{y}$. Then we claim that for almost all $s \in S$ and all $\gamma > -\mathbf{y}^1(s)$,

$$(38.4) \quad u(\mathbf{y}(s) + \gamma e_1, s) - u(\mathbf{y}(s), s) \leq \min (\alpha\gamma, \beta\gamma).$$

Indeed, if this is not so, then for each s in a subset U of positive measure, there is a $\boldsymbol{\gamma}(s)$ such that

$$u(\mathbf{y}(s) + \boldsymbol{\gamma}(s)e_1, s) - u(\mathbf{y}(s), s) > \min (\alpha\boldsymbol{\gamma}(s), \beta\boldsymbol{\gamma}(s)).$$

By the measurable choice theorem (Proposition 36.5), we may assume that $\boldsymbol{\gamma}(s)$ is measurable, and clearly it may be chosen integrable. Furthermore, either $\boldsymbol{\gamma}(s) > 0$ in a set of positive measure, or $\boldsymbol{\gamma}(s) < 0$ in a set of positive measure. In the first case, let V be that set, define $\mathbf{z}(s) = \mathbf{y}(s) + \boldsymbol{\gamma}(s)e_1$ for $s \in V$, $\mathbf{z}(s) = \mathbf{y}(s)$ otherwise. Setting $c = \int_S \mathbf{z}$ and $\gamma = \int_S \boldsymbol{\gamma}$, note that $\int_S u(\mathbf{z}) \leq u_S(c)$ and that $c = b + \gamma e_1$; hence

$$\begin{aligned} u_S(b + \gamma e_1) - u_S(b) &= u_S(c) - u_S(b) \geq \int_S (u(\mathbf{z}) - u(\mathbf{y})) \\ &= \int_V (u(\mathbf{z}) - u(\mathbf{y})) > \alpha\gamma, \end{aligned}$$

contradicting (38.2). In the second case (when $\underset{\sim}{y}(s) < 0$ in a set of positive measure), a contradiction is similarly obtained, using β instead of α. This establishes (38.4).

Since $\int_S \underset{\sim}{y}^1 = b^1 > 0$, there must be some set of s of positive measure in which $\underset{\sim}{y}^1(s) > 0$. Now from (38.4) it follows that for almost all $s \in S$ with $\underset{\sim}{y}^1(s) > 0$, the right-hand derivative of u_s w.r.t. x^1 at $x = \underset{\sim}{y}(s)$ is $\leq \alpha$, and the left-hand derivative is $\geq \beta$. Since $\beta > \alpha$, these two derivatives are unequal. So u_s is not differentiable at $\underset{\sim}{y}^1(s)$ w.r.t. x^1, contrary to $u \in \mathcal{U}_1$. This proves Proposition 38.1.

If f is a function differentiable at a point of E^n, we will denote by f' the vector $(f^1, \ldots, f^n)$ of its partial derivatives. In particular,

$$u_S' = (u_S^1, \ldots, u_S^n).$$

<u>PROPOSITION 38.5</u>. <u>Let</u> $u \in \mathcal{U}_1$, <u>let</u> $b \in \Omega$ <u>be</u> > 0, <u>and let</u> $u_S(b)$ <u>be attained at</u> $\underset{\sim}{y}$. <u>Then for all</u> $S \in \mathcal{C}$, <u>all</u> $j = 1, \ldots, n$, <u>and almost all</u> $s \in S$ <u>we have</u> $u_S^j(b) = u^j(\underset{\sim}{y}(s), s)$ <u>when</u> $\underset{\sim}{y}^j(s) > 0$. <u>Furthermore, for all</u> $x \in \Omega$ <u>and almost all</u> $s \in S$ <u>we have</u>

$$u(x, s) - u(\underset{\sim}{y}(s), s) \leqq u_S'(b) \cdot (x - \underset{\sim}{y}(s)).$$

Proof. By Proposition 36.4, there is a vector $p > 0$ such that

$$(38.6) \qquad u(x, s) - u(\underset{\sim}{y}(s), s) \leqq p \cdot (x - \underset{\sim}{y}(s))$$

for all $x \in \Omega$ and almost all s in S; furthermore, $p^j = u_s^j(\underset{\sim}{y}(s))$ for almost all s for which $\underset{\sim}{y}^j(s) > 0$.

Now for an arbitrary $\gamma \geqq -b^j$, let $u_S(b + \gamma e_j)$ be attained at $\underset{\sim}{z}$. Then by (38.6), a.e.

$$u(\underset{\sim}{z}(s), s) - u(\underset{\sim}{y}(s), s) \leqq p \cdot (\underset{\sim}{z}(s) - \underset{\sim}{y}(s)).$$

Integrating this inequality over S, we obtain

$$u_S(b + \gamma e_j) - u_S(b) \leqq p \cdot (b + \gamma e_j - b) = \gamma p^j.$$

By Proposition 38.1, the partial derivative u_S^j exists. Letting $\gamma \to 0+$, we deduce $u_S^j(b) \leqq p^j$; letting $\gamma \to 0-$, we deduce $u_S^j(b) \geqq p^j$. Hence $u_S^j(b) = p^j$ and Proposition 38.5 is proved.

The next proposition asserts that the gradient $u_S'(b)$ is continuous in b, and that this continuity has certain uniformity properties, both in b and in S.

For $\epsilon > 0$ and $\alpha > 0$, we denote

$$A(\epsilon, \alpha) = \{x \in \Omega : x \geq \epsilon e \quad \text{and} \quad \Sigma x \leq \alpha\}.$$

<u>PROPOSITION 38.7</u>. <u>Let</u> $u \in \mathcal{U}_1$. <u>Then for every</u> $\epsilon > 0$ <u>and every</u> $\alpha > 0$ <u>there is a</u> $\delta > 0$ <u>such that for all</u> S <u>with</u> $\mu(S) \geq \epsilon$ <u>and all</u> b <u>and</u> c <u>in</u> $A(\epsilon, \alpha)$ <u>with</u> $\|b - c\| \leq \delta$, <u>we have</u>

$$\|u_S'(b) - u_S'(c)\| < \epsilon.$$

<u>Outline of Proof</u>. It is not difficult to prove that a function that is concave and possesses all its partial derivatives at every point in the interior of Ω is necessarily continuously differentiable there (cf. Proposition 39.1). The function u_S satisfies these conditions (Propositions 36.3 and 38.1), and so it is continuously differentiable in the interior of Ω; since $A(\epsilon, \alpha)$ is compact, the continuity must be uniform w.r.t. b in $A(\epsilon, \alpha)$. Unfortunately, this line of argument will not yield the uniformity of the continuity w.r.t. S, which is essential for the applications in Section 40. We must therefore use a different attack.

Let b and c in $A(\epsilon, \alpha)$ be close to each other and let $u_S(b)$ and $u_S(c)$ be attained at $\underset{\sim}{y}$ and $\underset{\sim}{z}$ respectively.* Let

$$\underset{\sim}{\psi}(s) = (u_S'(c) - u_S'(b)) \cdot (\underset{\sim}{y}(s) - \underset{\sim}{z}(s));$$

from Proposition 38.5 it follows that $\underset{\sim}{\psi}$ is nonnegative. Since

$$\int \underset{\sim}{\psi} = (u_S'(c) - u_S'(b)) \cdot (b - c),$$

it follows that $\int \underset{\sim}{\psi}$ is small. But since $\underset{\sim}{\psi}$ is nonnegative, it follows that $\underset{\sim}{\psi}(s)$ itself is usually** small.

Suppose now that the conclusion of the proposition is false, i.e., that for some j, $u_S^j(b)$ and $u_S^j(c)$ are not close; say $u_S^j(b)$ is considerably larger than $u_S^j(c)$. Since $\underset{\sim}{\psi}(s)$ is usually small, it follows that $\underset{\sim}{y}^j(s)$ is usually close to $\underset{\sim}{z}^j(s)$. Furthermore, from Lemma 37.8 we know that $\underset{\sim}{y}^j$ and $\underset{\sim}{z}^j$ are bounded by some $\underset{\sim}{\eta}$, so they cannot usually vanish; it follows that there must be some s for which $\underset{\sim}{z}^j(s)$ is not close to 0, and moreover, $\underset{\sim}{y}^j(s)$ and $\underset{\sim}{z}^j(s)$ are close. If

*If it could be shown that for some s, $\underset{\sim}{y}(s)$ and $\underset{\sim}{z}(s)$ are close to each other and neither almost vanishes, then our result would follow from the continuous differentiability of $u(\cdot, s)$ and Proposition 38.5. But this is not necessarily true.

**I.e., for all s except for a set of small measure.

we now proceed in the positive x^j-direction from $z^j(s)$, then near $z(s)$, u will be rising at a rate given approximately* by

$$\frac{\partial}{\partial x^j} u(x, s)|_{x=z(s)};$$

by Proposition 38.5, this is equal to $u_S^j(c)$. On the other hand, if k is a coordinate for which $y^k(s)$ differs considerably** from $z^k(s)$, then since $y(s)$ is small, $u_S^k(b)$ must be close to $u_S^k(c)$. Therefore along the line connecting $z(s)$ with $y(s)$, the hyperplanes H_b and H_c given respectively by

$$u = u(y(s), s) + u_S'(b)(x - y(s))$$

and

$$u = u(z(s), s) + u_S'(c)(x - z(s))$$

must be almost parallel. But these hyperplanes support the graph of $u(\cdot, s)$ (Proposition 38.5) and pass through it at the points corresponding to $y(s)$ and $z(s)$ respectively.

*Because $u(\cdot, s)$ is continuously differentiable.

**If there is such a coordinate If not, the $y(s)$ is close to $z(s)$, and so in the argument below, H_b automatically passes close to the graph of $u(\cdot, s)$ at $z(s)$.

Therefore H_b and H_c almost coincide along the line from $\underset{\sim}{y}(s)$ to $\underset{\sim}{z}(s)$, and in particular, H_b passes close to the graph of $u(\cdot, s)$ at $\underset{\sim}{z}(s)$. But since H_b supports the graph of $u(\cdot, s)$, it then follows that the rate of rise of u in the positive x^j-direction from $\underset{\sim}{z}(s)$ cannot be much greater than $u_S^j(b)$, at least if we average over a large enough x^j-interval. But this is in contradiction to the fact that this rate is approximately $u_S^j(c)$, as shown above.

Proof of Proposition 38.7. Let $\underset{\sim}{\eta}$ correspond to ϵ and α in accordance with Lemma 37.8, let b and c be in $A(\epsilon, \alpha)$, and let $u_S(b)$ and $u_S(c)$ be attained at $\underset{\sim}{y}$ and $\underset{\sim}{z}$ respectively. We first wish to prove that there is a number β, depending on u, ϵ, and α only (and not on the choices of S, b or c), such that

$$(38.8) \qquad \Sigma u_S'(b) \leqq \beta \quad \text{and} \quad \Sigma u_S'(c) \leqq \beta .$$

Indeed, setting $x = 0$ in Proposition 38.5, we obtain

$$u_S'(b) \cdot \underset{\sim}{y}(s) \leqq u(\underset{\sim}{y}(s), s) \leqq u(\underset{\sim}{\eta}(s)e, s).$$

Integrating over S, we obtain

$$u_S'(b)\cdot\int_S \underset{\sim}{y} \leqq \int_S u(\underset{\sim}{\eta}e) \leqq \int u(\underset{\sim}{\eta}e).$$

Since $b \in A(\epsilon, \alpha)$, we have $b \geqq \epsilon e$; therefore

$$\epsilon\Sigma u_S'(b) = u_S'(b)\cdot\epsilon e \leqq u_S'(b)\cdot b = u_S'(b)\cdot\int_S \underset{\sim}{y} \leqq \int u(\underset{\sim}{\eta}e).$$

Thus $\Sigma u_S'(b) \leqq \int u(\underset{\sim}{\eta}e)/\epsilon$, and similarly $\Sigma u_S'(c) \leqq \int u(\underset{\sim}{\eta}e)/\epsilon$. Setting $\beta = \int u(\underset{\sim}{\eta}e)/\epsilon$, we deduce (38.8).

In the remainder of the proof, let j be a fixed index. We next claim that there is a number $\delta_1 > 0$ (depending on u, ϵ, α, and j), such that

$$(38.9)\qquad \begin{cases} \mu\{s \in S : \underset{\sim}{y}^j(s) > \delta_1\} > \delta_1, & \text{and} \\ \\ \mu\{s \in S : \underset{\sim}{z}^j(s) > \delta_1\} > \delta_1. \end{cases}$$

Indeed, for a fixed δ_1, let $U = \{s \in S : \underset{\sim}{y}^j(s) > \delta_1\}$. If $\mu(U) \leqq \delta_1$, then

$$\int_S \underset{\sim}{y}^j \leq \int_U + \int_{S\setminus U} \leq \int_U \underset{\sim}{\eta} + \delta_1\mu(S\setminus U).$$

Since $\underset{\sim}{\eta}$ is integrable, it follows that the right side of this inequality will be $< \epsilon$ if δ_1 is chosen sufficiently small; but this contradicts $\epsilon \leq b^j = \int_S \underset{\sim}{y}^j$. The same reasoning applies to $\underset{\sim}{z}$. This proves (38.9).

Because $u^j(x, s)$ is continuous in x for each fixed s whenever $x^j > 0$, it is, for each fixed s, uniformly continuous in the set

$$\underset{\sim}{C}(s) = \{x \in \Omega : x^j \geq \delta_1, \quad x \leq \underset{\sim}{\eta}(s)e\}.$$

So for each fixed s we may find a number $\underset{\sim}{\delta}_2(s) > 0$ such that

$$|u^j(y, s) - u^j(z, s)| < \epsilon/3$$

whenever $\|y - z\| \leq \underset{\sim}{\delta}_2(s)$ and $y, z \in \underset{\sim}{C}(s)$. Furthermore, it may be shown that $\underset{\sim}{\delta}_2$ may be chosen measurable, and we may assume w.l.o.g. that

$$\underset{\sim}{\delta}_2(s) \leq 1$$

for all s. Since $\underset{\sim}{\delta}_2$ is measurable and bounded, it is integrable. Since it is always positive, it follows that if we define

$$\delta_3 = \inf \{\int_U \underset{\sim}{\delta}_2 : \mu(U) \geqq \delta_1/2\},$$

then

$$\delta_3 > 0.$$

Finally, choose δ so that

$$\delta < \varepsilon\delta_3/3n\beta.$$

Let

$$\underset{\sim}{v}(s) = (u_S'(c) - u_S'(b)) \cdot (\underset{\sim}{y}(s) - \underset{\sim}{z}(s)).$$

From Proposition 38.5 it follows that for $s \in S$,

$$\begin{aligned} u_S'(b) \cdot (\underset{\sim}{y}(s) - \underset{\sim}{z}(s)) &\leqq u(\underset{\sim}{y}(s), s) - u(\underset{\sim}{z}(s), s) \\ &\leqq u_S'(c) \cdot (\underset{\sim}{y}(s) - \underset{\sim}{z}(s)); \end{aligned}$$

hence

$$\text{(38.10)} \qquad \underset{\sim}{\psi}(s) \geqq 0.$$

Hence by the definition of β,

$$\begin{aligned}\text{(38.11)} \quad 0 &\leqq \int_S \underset{\sim}{\psi} = (u_S'(c) - u_S'(b)) \cdot \int(\underset{\sim}{y} - \underset{\sim}{z}) \\ &= (u_S'(c) - u_S'(b)) \cdot (b - c) \leqq \|u_S'(c) - u_S'(b)\| n \| b - c \| \\ &\leqq \beta n \delta < \epsilon\delta_3/3 < \epsilon\delta_3/2.\end{aligned}$$

Setting

$$W = \{s \in S : \underset{\sim}{\psi}(s) \geqq \epsilon\underset{\sim}{\delta}_2(s)/2\},$$

we obtain

$$\text{(38.12)} \qquad \mu(W) < \delta_1/2;$$

for if $\mu(W) \geqq \delta_1/2$, then because $\underset{\sim}{\psi}$ is nonnegative ((38.10)),

$$\int_S \underset{\sim}{\psi} \geqq \int_W \underset{\sim}{\psi} \geqq \frac{\epsilon}{2}\int_W \underset{\sim}{\delta}_2 \geqq \epsilon\delta_3/2,$$

contradicting (38.11).

Suppose now that the proposition is false, i.e. that

$$|u_S^j(b) - u_S^j(c)| > \varepsilon$$

for some j, w.l.o.g. the one we have fixed. Assume first that $u_S^j(c) - u_S^j(b) > \varepsilon$. From (38.12) and (38.9) it follows that there must be an s in S such that $\underset{\sim}{z}^j(s) > \delta_1$ and $s \notin W$. Choose such an s. Since $s \notin W$, we deduce that $\underset{\sim}{\gamma}(s) < \varepsilon\underset{\sim}{\delta}_2(s)/2$. Set $\gamma = \underset{\sim}{\gamma}(s)$, $\delta_2 = \underset{\sim}{\delta}_2(s)$, $y = \underset{\sim}{y}(s)$, $z = \underset{\sim}{z}(s)$, $w = z + \frac{\delta_2}{2}e_j$, $w^\# = z + \delta_2 e_j$. Then

$$(u_S'(c) - u_S'(b))\cdot(y - w) = \gamma - (u_S^j(c) - u_S^j(b))\frac{\delta_2}{2} < \frac{\varepsilon\delta_2}{2} - \frac{\varepsilon\delta_2}{2} = 0.$$

Hence

$$u_S'(b)\cdot(w - y) < u_S'(c)\cdot(w - y).$$

Hence by Proposition 38.5,

$$u(w^\#, s) - u(y, s) \leqq u_S'(b)\cdot(w^\# - y) = u_S'(b)(w + \frac{\delta_2}{2}e_j - y)$$

$$= u_S'(b)\cdot(w - y) + {}_S^j(b)\frac{\delta_2}{2} < u_S'(c)(w - y) + u_S^j(b)\frac{\delta_2}{2}.$$

On the other hand, for an appropriate $\theta \in [0, 1]$,

$$u(w^{\#}, s) - u(y, s) = (u(w^{\#}, s) - u(z, s)) - (u(y, s) - u(z, s))$$

$$= u^j(z + \theta\delta_2 e_j, s)\delta_2 - (u(y, s) - u(z, s))$$

$$\geq (u^j(z, s) - \tfrac{\epsilon}{3})\delta_2 - u_S'(c)\cdot(y - z),$$

because of the definition of $\delta_2 = \underset{\sim}{\delta}_2(s)$ and Proposition 38.5. Again using Proposition 38.5 and $z^j = \underset{\sim}{z}^j(s) > 0$, we deduce that

$$u(w^{\#}, s) - u(y, s) \geq u_S^j(c)\delta_2 - \frac{\epsilon\delta_2}{3} - u_S'(c)\cdot(y - z)$$

$$= u_S^j(c)\frac{\delta_2}{2} - \frac{\epsilon\delta_2}{3} + u_S'(c)\cdot(w - y).$$

Combining the two inequalities for $u(w^{\#}, s) - u(y, s)$, we deduce that

$$u_S^j(b)\frac{\delta_2}{2} > u_S^j(c)\frac{\delta_2}{2} - \epsilon\frac{\delta_2}{3}.$$

Hence

$$\epsilon\frac{\delta_2}{3} > (u_S^j(c) - u_S^j(b))\frac{\delta_2}{2} > \epsilon\frac{\delta_2}{2}.$$

Hence 2 > 3, an absurdity. The case $u_S^j(b) - u_S^j(c) > \epsilon$ is handled similarly. This completes the proof of Proposition 38.7.

<u>COROLLARY 38.13</u>. <u>Let</u> $u \in \mathcal{U}_1$. <u>Then for all</u> $j = 1, \ldots, n$ <u>and all</u> $S \in \mathcal{C}$, u_S' <u>is continuous at each</u> b <u>in</u> Ω <u>with</u> $b > 0$.

<u>PROPOSITION 38.14</u>. <u>Let</u> $u \in \mathcal{U}_1$. <u>Then for every</u> $\epsilon > 0$ <u>and every</u> $\alpha > 0$ <u>there is a</u> $\delta > 0$ <u>such that for all</u> S <u>with</u> $\mu(S) \geq \epsilon$, <u>all</u> b <u>in</u> $A(\epsilon, \alpha)$, <u>and all</u> δ<u>-approximations</u> $\hat{u}$ <u>to</u> u, <u>we have</u>

$$\|\hat{u}_S'(b) - u_S'(b)\| < \epsilon.$$

<u>Proof</u>. Fix j. Let $\epsilon_1 = \frac{1}{2}\epsilon$, and let δ_1 correspond to ϵ_1 and $\alpha + \epsilon_1$ in accordance with Proposition 38.7; furthermore, choose $\delta_1 < \epsilon_1$. Let δ correspond to $\delta_1\epsilon/4(1 + \delta_1 + \alpha)$ in accordance with Proposition 37.11. Then since $\hat{u}$ is a δ-approximation to u, we have

$$|\hat{u}_S(b) - u_S(b)| < \delta_1 \epsilon/4$$

and

$$|\hat{u}(b + \delta_1 e_j) - u_S(b + \delta_1 e_j)| < \delta_1 \epsilon/4.$$

Hence

$$\left|\frac{\hat{u}_S(b + \delta_1 e_j) - \hat{u}_S(b)}{\delta_1} - \frac{u_S(b + \delta_1 e_j) - u_S(b)}{\delta_1}\right| < \frac{\epsilon}{2}.$$

Similarly

$$\left|\frac{\hat{u}_S(b) - \hat{u}_S(b - \delta_1 e_j)}{\delta_1} - \frac{u_S(b) - u_S(b - \delta_1 e_j)}{\delta_1}\right| < \frac{\epsilon}{2} .$$

But because of the concavity of $\hat{u}_S$ (Proposition 36.3), we have

$$\frac{\hat{u}_S(b) - \hat{u}_S(b - \delta_1 e_j)}{\delta_1} \geq \hat{u}_S^j(b) \geq \frac{\hat{u}_S(b + \delta_1 e_j) - \hat{u}_S(b)}{\delta_1}.$$

Thus, again because of the concavity,

$$\hat{u}_S^j(b) \geq \frac{u_S(b) + \delta_1 e_j) - u_S(b)}{\delta_1} - \frac{\epsilon}{2}$$

$$\geq u_S^j(b + \delta_1 e_j) - \frac{\epsilon}{2}$$

$$\geq u_S^j(b) - \frac{\epsilon}{2} - \frac{\epsilon}{2} = u_S^j(b) - \epsilon .$$

Similarly $\hat{u}_S^j(b) \leq u_S^j(b) + \epsilon$. This completes the proof of Proposition 38.14.

39. THE FINITE TYPE CASE

In this section we shall prove* Proposition 31.5.

PROPOSITION 39.1. Let f be a continuous concave function defined on Ω, and for some j with $1 \leq j \leq n$, assume that $f^j = \partial f/\partial x^j$ exists at each $x \in \Omega$ such that $x^j > 0$. Then f^j is continuous at each point x such that $x^j > 0$.

Proof. Without loss of generality let $j = 1$. Suppose f^1 is not continuous, say at y, where $y^1 > 0$. Let $x_k \to y$, $f^1(x_k) \to \alpha$, $\alpha \neq f^1(y)$ (possibly $\alpha = \pm\infty$). Without loss of generality let $x_k^1 > y^1/2$ for all k. Then for all $\gamma > -y^1/2$, we have

$$(39.2) \qquad f(x_k + \gamma e_1) - f(x_k) \leq f^1(x_k)\gamma.$$

If α is finite, let $k \to \infty$ and obtain from the continuity of f that

$$f(y + \gamma e_1) - f(y) \leq \alpha\gamma$$

for all $\gamma > -y^1/2$. Hence, because $f^1(y)$ exists it must be equal to α, a contradiction. If $\alpha = \pm\infty$, let $\gamma = \mp y^1/4$; then

*Very few of the tools developed in Sections 35, 37, and 38 will be used in the process, and sometimes only special cases--which could have been established more easily than the general cases--will be used. All in all, what is needed from those sections for the finite type case could have been developed separately in a few pages. We did not do this because we wished to avoid an unnecessary duplication, and because we consider the finite type case to be chiefly a stepping stone to the general one.

(39.2) yields $f(x_k + \gamma e_1) - f(x_k) \to -\infty$, whereas continuity implies that it tends to $f(y + \gamma e_1) - f(y)$. This contradiction proves Proposition 39.1.

PROPOSITION 40.3. Let $f \in \mathfrak{F}_0$, and let $u \in \mathcal{U}_0$ be defined by $u_s = f$ for all $s \in I$. Then for all $a \in \Omega$ we have

$$u_I(a) = f^*(a),$$

where f^* is the concavification* of f.

Proof. Assume first that $a > 0$ and that f is concave. Let

$$G = \{(\nu, x) \in E^1 \times \Omega : \nu \leqq f(x)\}.$$

Then G is concave, and $(f(a), a)$ is on the boundary of G. So there is a hyperplane containing $(f(a), a)$ that supports G, i.e., there is a $q \in E^n$ and a $\rho \in E^1$ such that $(\rho, q) \neq 0$ and

*See Section 36, in particular Proposition 36.2.

(39.4) $\rho \cdot \nu - q \cdot x \leqq \rho \cdot f(a) - q \cdot a$

for all $(\nu, x) \in G$. If for any j, we would have $q^j < 0$, then by setting $\nu = 0$ and letting x^j be large and $x^i = 0$ for $i \neq j$, we would get a contradiction to (39.4). Hence, $q \in \Omega$. If $\rho < 0$, then by fixing x and letting ν be a negative number with large absolute value, we again get a contradiction to (39.4). If $\rho = 0$ then by $(\rho, q) \neq 0$ and $q \in \Omega$ we get $q \geq 0$; hence since $a > 0$, we get $q \cdot a > 0$. But if we set $\nu = 0$ and $x = 0$ in (39.4), we get $0 \leqq -q \cdot a$, which is again a contradiction. We conclude that $\rho > 0$, which permits us to divide (39.4) by ρ and obtain a $p \in \Omega$ such that $\nu - p \cdot x \leqq f(a) - p \cdot a$ for all $(\nu, x) \in G$. If in particular we set $\nu = f(x)$ and recall that $u(x, s) = f(x)$ for all s, we deduce that

$$u(x, s) - u(a, s) \leqq q \cdot (x - a).$$

Hence by Proposition 36.4, $u_I(a)$ is attained at $\underset{\sim}{x} \equiv a$. Since f is concave, $f^* = f$, and so the proposition is proved in this case.

When f is concave but a is not necessarily > 0, then we apply the case just proved to the subspace of E^n obtained

by considering only those coordinates j for which a^j is positive, and obtain the desired result.

When f is not necessarily concave, then we apply Proposition 36.3, and deduce from the concave case just proved that

$$u_I(a) = u^*{}_I(a) = f^*(a).$$

This completes the proof of Proposition 39.3.

<u>LEMMA 39.5</u>. <u>Let</u> $f \in \mathfrak{F}_1$. <u>Then the concavification</u> f^* <u>is also in</u> $\mathfrak{F}_1$.

<u>Proof</u>. Define $u \in \mathcal{U}_1$ by $u(x, s) = f(x)$ for all $s \in S$. By Proposition 36.3, $u^*_s \in \mathcal{U}_0$ for all $s \in S$, and hence $f^* \in \mathfrak{F}_0$. To prove that f^* obeys the differentiability condition (35.2), note that by Proposition 39.3, $u_I(x) = f^*(x)$ for all $x \in \Omega$. Hence by Proposition 38.1, $f^{*j}(x)$ exists whenever $x^j > 0$. Since f^* is continuous and concave, it follows from Proposition 39.1 that f^{*j} is continuous at each $x \in \Omega$ for which $x^j > 0$, and so (35.2) is verified. Finally, the fact that f^* is increasing follows from Lemma 37.12. This completes the proof of Lemma 39.5.

<u>COROLLARY 39.6</u>. <u>If</u> $u \in \mathcal{U}_1$, <u>then</u> $u^* \in \mathcal{U}_1$.

<u>Proof</u>. This is an immediate consequence of Lemma 39.5 and Proposition 36.3.

Let $f_1, \ldots, f_k$ be concave members of $\mathfrak{F}_1$. Denote the nonnegative orthant of E^k by Ξ. For $y \in \Xi$ and $z \in \Omega$, define

$$(39.7)\quad g(y, z) = \max \{\Sigma_{i=1}^{k} y^i f_i(x_i) : x_1, \ldots, x_k \in \Omega \text{ and } \Sigma_{i=1}^{k} y^i x_i \leqq z\}.$$

If we set $w = (y, z)$, then $w \in \Xi \times \Omega \subset E^{n+k}$. Thus $g = g(w)$ is a function of $k + n$ nonnegative real variables.

Note that the inequality sign in the constraint $\Sigma_{i=1}^{k} y^i x_i \leqq z$ may be replaced by an equality unless $\Sigma_{i=1}^{k} y^i = 0 < \Sigma_{j=1}^{n} z^j$.

It is easily seen that the max in the definition of g is attained. Indeed, if $y > 0$, then the constraint set is compact; and if one or more of the coordinates of y vanish, then we can ignore those coordinates and the corresponding x_i entirely, and the constraint set for the remaining x_i will still be compact. If all the y^i vanish— i.e., if $y = 0$—then, of course, $g(y, z) = 0$, and the max is achieved for any k-tuple of x_i in Ω.

LEMMA 39.8. *Let* $\{S_1, \ldots, S_k\}$ *be a partition of* I, *and define* u *in* $\mathcal{U}_1$ *of finite type by*

$$u(x, s) = f_i(x) \quad \text{when } s \in S_i.$$

For $S \subset I$, *define* $y_S \in \mathfrak{a}$ *by* $y_S^i = \mu(S \cap S_i)$. *Then*

$$u_S(z) = g(y_S, z)$$

for all $z \in \Omega$.

Proof. Let $g(y_S, z)$ be attained at $(x_1, \ldots, x_k)$. Define $\underset{\sim}{x}$ by

$$\underset{\sim}{x}(s) = x_i \quad \text{for} \quad s \in S \cap S_i.$$

Then

$$\int_S \underset{\sim}{x} = \Sigma\, y_S^i x_i \leqq z,$$

and hence

$$u_S(z) \geq \int_S u(\underset{\sim}{x}) = \Sigma_{i=1}^k y_S^i f_i(x_i) = g(y_S, z).$$

To obtain the opposite inequality, let $u_S(z)$ be attained at $\underset{\sim}{x}$. Define $(x_1, \ldots, x_k)$ by

$$x_i = \begin{cases} \frac{1}{\mu(S \cap S_i)} \int_{S \cap S_i} \underset{\sim}{x}, & \text{if } \mu(S \cap S_i) \neq 0 \\ \text{arbitrary}, & \text{if } \mu(S \cap S_i) = 0. \end{cases}$$

Then by the concavity of f_i,

$$f_i(x_i) \geq \frac{1}{\mu(S \cap S_i)} \int_{S \cap S_i} f_i(\underset{\sim}{x}) = \frac{1}{y_S^i} \int_{S \cap S_i} u(\underset{\sim}{x})$$

if $\mu(S \cap S_i) \neq 0$. Furthermore

$$\Sigma_{i=1}^k y_S^i x_i = \Sigma_{i=1}^k \mu(S \cap S_i) x_i = \int_S \underset{\sim}{x} = z.$$

Hence

$$g(y_S, z) \geq \Sigma_{i=1}^k y_S^i f_i(x_i) = \Sigma_{i=1}^k \int_{S \cap S_i} u(\underset{\sim}{x}) = \int_S u(\underset{\sim}{x}) = u_S(z).$$

This completes the proof of the lemma.

LEMMA 39.9. g is concave, nondecreasing, and continuous on $\Xi \times \Omega$.

Proof. We first prove the concavity. Indeed, let (y_1, z_1) and (y_2, z_2) be in $\Xi \times \Omega$, and let $g(y_1, z_1)$ and $g(y_2, z_2)$ be taken on at $\{x_{11}, \ldots, x_{1k}\}$ and $\{x_{21}, \ldots, x_{2k}\}$ respectively. For $0 \leqq \alpha \leqq 1$, let

$$(y, z) = \alpha(y_1, z_1) + (1 - \alpha)(y_2, z_2),$$

and for each i define

$$x_i = \begin{cases} (\alpha y_1^i x_{1i} + (1 - \alpha) y_2^i x_{2i})/y^i, & \text{if } y^i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \Sigma_{i=1}^k y^i x_i &= \Sigma_{i=1}^k [\alpha y_1^i x_{1i} + (1 - \alpha) y_2^i x_{2i}] \\ &= \alpha \Sigma_{i=1}^k y_1^i x_{1i} + (1 - \alpha) \Sigma_{i=1}^k y_2^i x_{2i} \\ &\leqq \alpha z_1 + (1 - \alpha) z_2 \\ &= z. \end{aligned}$$

So if we let $L = \{i : 1 \leqq i \leqq k \text{ and } y^i > 0\}$, we obtain from the definition of g and the concavity of the f^i that

$$g(y, z) \geqq \Sigma_{i=1}^{k} y^i f_i(x_i)$$

$$\geqq \Sigma_{i \in L} y^i \left[\frac{\alpha y_1^i}{y^i} f_i(x_{1i}) + \frac{(1-\alpha) y_2^i}{y^i} f_i(x_{2i}) \right]$$

$$= \alpha \, \Sigma_{i=1}^{k} y_1^i f_i(x_{1i}) + (1-\alpha) \, \Sigma_{i=1}^{k} y_2^i f_i(x_{2i})$$

$$= \alpha g(y_1, z_1) + (1-\alpha) g(y_2, z_2).$$

This shows that g is indeed concave.

Next, we show that g is nondecreasing in $w = (y, z)$. Indeed, suppose $w_1 \geq w_0$ and $g(w_1) < g(w_0)$. If we draw a straight ray (half-line) starting at w_0 and passing through w_1, then this ray must always stay in $\Xi \times \Omega$. On the other hand, from the concavity of g it follows that at a point on the ray sufficiently beyond w_1, g will be negative. But it is clear from its definition that g can never be negative. This demonstrates that g is nondecreasing.

Finally, we prove the continuity of g at each point $w_0 = (y_0, z_0)$ of $\Xi \times \Omega$. Note first that g is homogeneous of degree 1. Hence we may wholly restrict ourselves to the case in which $\Sigma_{i=1}^{k} y_0^i \leqq \frac{1}{2}$. In that case we may find a partition $\{S_1, \ldots, S_k\}$ of I and an $S \subset I$ such that

$$y_0^i = \mu(S \cap S_i).$$

If we define u in $\mathcal{U}_1$ as in Lemma 39.8, then from that lemma it follows that

$$u_S(z) = g(y_0, z)$$

for all z in Ω. Hence from Proposition 37.13 it follows that $g(y, z)$ is continuous in z at (y_0, z_0).

To complete the proof of continuity it is sufficient to demonstrate

(39.10) For every ϵ there is a δ such that if $\|(y, z) - (y_0, z_0)\| \leqq \delta$, then $|g(y, z) - g(y_0, z)| < \epsilon$.

So let $\epsilon > 0$ be given. For each i there is an η such that $f_i(x) \leqq \epsilon\|x\|$ whenever $\|x\| > \eta$; w.l.o.g. we may choose the same η for all i. It then follows* that for all x and i, we have

(39.11) $$f_i(x) < \epsilon(\eta + \Sigma x).$$

*Compare the proof of Lemma 37.9.

Now for given (y, z), define $\hat{y} \in \Xi$ by $\hat{y}^i = \min(y^i, y_0^i)$ for all i. Let $g(y, z)$ be attained at $(x_1, \ldots, x_k)$. Since $\hat{y} \leqq y$, it follows that $(x_1, \ldots, x_k)$ satisfies the constraints in the definition of $g(\hat{y}, z)$. Hence

$$g(\hat{y}, z) \geqq \Sigma_{i=1}^k \hat{y}_i f_i(x_i).$$

Therefore, using the monotonicity of g and (40.11),

$$\begin{aligned}
(39.12) \qquad & g(y, z) - g(\hat{y}, z) \\
& \leqq \Sigma_{i=1}^k y^i f_i(x_i) - \Sigma_{i=1}^k \hat{y}^i f_i(x_i) \\
& \leqq \epsilon\eta\, \Sigma_{i=1}^k (y^i - \hat{y}^i) + \epsilon\, \Sigma_{j=1}^n \Sigma_{i=1}^k (y^i - \hat{y}^i) x_i^j \\
& \leqq \epsilon\eta\, \Sigma_{i=1}^k (y^i - \hat{y}^i) + \epsilon\, \Sigma_{j=1}^n \Sigma_{i=1}^k y^i x_i^j \\
& = \epsilon\eta\, \Sigma_{i=1}^k (y^i - \hat{y}^i) + \epsilon\, \Sigma z.
\end{aligned}$$

Now from the definition of $\hat{y}$ it follows that

$$\|y - \hat{y}\| \leqq \|y - y_0\| \leqq \|(y, z) - (y_0, z_0)\|;$$

hence if in (39.10), δ is chosen sufficiently small, then the first term on the right side of (39.12) may be made less than ϵ, say. As for the second term, if δ is chosen

less than 1/n, say, then we will have

$$\epsilon\Sigma\, z \leqq \epsilon(1 + \Sigma z_0).$$

Thus altogether we obtain

$$g(y, z) - g(\hat{y}, z) \leqq \epsilon(2 + \Sigma z_0).$$

Since g is monotonic and $y \geqq \hat{y}$, it follows that $g(y, z) - g(\hat{y}, z) \geqq 0$. Hence

$$\|g(y, z) - g(\hat{y}, z)\| \leqq \epsilon(2 + \Sigma z_0).$$

Similarly we obtain

$$\|g(y_0, z) - g(\hat{y}, z)\| \leqq \epsilon(2 + \Sigma z_0).$$

Hence

$$\|g(y, z) - g(y_0, z)\| \leqq \epsilon(4 + 2\Sigma z_0).$$

This gives us (39.10) with a factor of $\epsilon(4 + 2\Sigma z_0)$,

and (39.10) follows without difficulty. Thus the proof of continuity is complete, and with it the proof of Lemma 39.9.

<u>PROPOSITION 39.13</u>. <u>For each</u> p <u>with</u> $1 \leqq p \leqq k + n$, $g^p = \partial g/\partial w^p$ <u>exists and is continuous at each</u> $w \in \Xi \times \Omega$ <u>for which</u> $w^p > 0$.

<u>Proof</u>. First, we show that g is differentiable in y^1 whenever $y^1 > 0$. Indeed, let $y_0 \in \Xi$, $z_0 \in \Omega$, and let $g(y_0, z_0)$ be taken on at $\{x_{01}, \ldots, x_{0k}\}$. Define a function h of the positive real variable y^1 by

$$h(y^1) = y^1 f_1(x_{01}y_0^1/y^1) + \Sigma_{i=2}^k y_0^i f_i(x_{0i}).$$

Since we wish to fix attention on the variable y^1, it is convenient to set $g_1(y^1) = g(y^1, y_0^2, \ldots, y_0^k, z_0)$; in particular, therefore, $g_1(y_0^1) = g(y_0, z_0)$. Now since

$$y^1(x_{01}y_0^1/y_1) + \Sigma_{i=2}^k y_0^i x_{0i} = \Sigma_{i=1}^k y_0^i x_{0i} = z,$$

if follows from the definitions of g and g_1 that

$$h(y^1) \leqq g_1(y^1) \tag{39.14}$$

for all $y^1 > 0$, and of course

$$h(y_0^1) = g_1(y_0^1). \tag{39.15}$$

Next, note that h is differentiable whenever $y^1 > 0$; this follows from the differentiability of $f_1(x_{01}y_0^1/y^1)$ as a function of y^1, which in turn follows from the fact that $f^j(x)$ exists whenever $x^j > 0$. Of course it may happen that some of the coordinates of x_{01} vanish, but then any change in y^1 does not affect the corresponding coordinates of $x_{01}y_0^1/y^1$, so that the differentiability of h is not affected. In fact, if we let $M = \{j \in N : x_{01}^j > 0\}$, then

$$\begin{aligned} h'(y^1) &= \frac{d}{dy^1} h(y^1) \\ &= f_1(x_{01}y_0^1/y^1) + y^1 \, \Sigma_{j \in M} \, f_1^j(x_{01}y_0^1/y_1)\left(-\frac{x_{01}^j y_0^1}{(y^1)^2}\right). \end{aligned}$$

In particular, we obtain

$$h'(y_0^1) = f_1(x_{01}) - \Sigma_{j \in M} \, x_{01}^j f_1^j(x_{01}).$$

On the other hand, since g is concave in w, it follows that g_1 is concave in y^1; hence there is a supporting line to its graph at y_0^1, i.e., there is a linear function $\ell(y^1)$ such that

$$\ell(y^1) \geq g_1(y^1)$$

for all $y^1 > 0$, and

$$\ell(y_0^1) = g_1(y_0^1).$$

Recalling (39.14) and (39.15), we find that g_1 is "trapped" between the two differentiable functions ℓ and h at y_0^1, and so must be differentiable. The differentiability of g_1 at y_0^1 is of course the same thing as the existence of the derivative $\partial g/\partial y^1$ at (y_0, z_0). A similar argument shows that all the derivatives $\partial g/\partial y^i$ exist whenever $y^i > 0$, for each i in $\{1, \ldots, k\}$.

The existence of the partial derivatives $\partial g/\partial z^j$ for $z^j > 0$ is an easy consequence of Proposition 38.1 and Lemma 39.8.

Combining the existence of the partial derivatives with the continuity and concavity of g (Lemma 39.9), and

applying Proposition 39.1, we deduce the required continuity of the derivatives.*

Denote by H the set of all superadditive set-functions in pNA that are homogeneous of degree 1 (see Part IV).

LEMMA 39.16. *If* $u \in \mathcal{U}_1$ *and* u *is of finite type, then* v *(as defined in* (30.1)) *is in* H.

Proof. First we show that $v \in$ pNA. By Corollary 39.6, we have $u^* \in \mathcal{U}_1$, and certainly u^* is of finite type as well. Thus there is a finite set $\{f_1, \ldots, f_k\}$ of concave functions in $\mathfrak{F}_1$ such that each u_S^* is one of the f_i. If we now define g by (39.7), then from Lemma 39.8 we obtain $u_S^*(z) = g(y_S, z)$ for all $S \subset I$ and $z \in \Omega$, where y_S is defined by

$$y_S^i = \mu(S \cap S_i)$$

*The basic idea of this proof, to prove the differentiability of a function by "trapping" it between two differentiable functions, was adapted from [S_8] (see the lemma on p. 7, and its proof on pp. 8–9 of [S_8]).

and S_i is defined by

$$S_i = \{s \in I : u_s^* = f_i\}.$$

Now by Proposition 36.3, $u_S^*(z) = u_S(z)$; hence

$$u_S(z) = g(y_S, z).$$

If we write $\eta(S)$ instead of y_S, then we see that η is a k-dimensional vector of non-atomic measures on I, and we have

$$(39.17) \qquad u_S(z) = g(\eta(S), z).$$

Now define an n-dimensional vector ζ of NA-measures by

$$\zeta(S) = \int_S \underset{\sim}{a}.$$

Substituting $\zeta(S)$ for z in (40.17) and using the definition of v (30.1), we obtain

$$v(S) = u_S(\zeta(S)) = g(\eta(S), \zeta(S)).$$

Letting $\nu = (\eta, \zeta)$, we see that ν is a vector of nonnegative measures in NA, and that

$$v = g \circ \nu. \tag{39.18}$$

Since g is continuous and nondecreasing (Lemma 39.9), and since for all i, $\partial g/\partial w_i$ exists and is continuous whenever $w^i > 0$, it follows from Proposition 9.17 that $g \circ \nu$, and hence v, is in pNA.

To show that v is homogeneous of degree 1, use the Weierstrass approximation theorem in k + n dimensions to find a sequence $\{h_j\}$ of polynomials (in k + n variables) such that

$$|h_j(w) - g(w)| \leqq 1/j$$

for all w in the range of ν. For these polynomials it follows from the defining properties of the extension operator (in particular (21.1), (21.2), and (21.3)) that

$$(h_j \circ \nu)^*(\alpha\chi_S) = h_j(\alpha\nu(S)),$$

where the * denotes the extension (see Part III). Letting

$j \to \infty$ and using the continuity of the extension operator in the supremum norm (22.9), we deduce that

$$v^*(\alpha\chi_S) = (g \circ \nu)^*(\alpha\chi_S) = g(\alpha\nu(S)).$$

But since it is easily verified that g is homogeneous of degree 1, it follows that

$$g(\alpha\nu(S)) = \alpha g(\nu(S)) = \alpha v(S).$$

Hence $v^*(\alpha\chi_S) = \alpha v(S)$, and so v is homogeneous of degree 1.

We have demonstrated that $v \in pNA$ and that it is homogeneous of degree 1. Since its superadditivity is obvious, the proof of Lemma 39.16 is complete.

We are now ready for the

Proof of Proposition 31.5. This follows immediately from Lemma 39.16 and Theorem F.

40. PROOF OF THEOREM G

The proof will proceed by reducing the general case to the finite type case. In the process, we shall also prove Proposition 33.2.

LEMMA 40.1. *Let* $u \in \mathcal{U}_1$ *and* $\underset{\sim}{a}$ *be given, where* $\underset{\sim}{a}$ *is* μ*-integrable. Then for every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that if* m *is a positive integer,* $\hat{u} \in \mathcal{U}_1$ *is a* δ*-approximation to* u*, and*

$$S_1 \subset \ldots \subset S_m \subset S_{m+1} = I$$

is a sequence such that $\int_{S_1} \underset{\sim}{a} \geq \epsilon e$ *and* $u(S_{k+1} \setminus S_k) \leq \delta$ *for all* k*, then*

$$(40.2)\quad \Sigma_{k=1}^{m} |v(S_{k+1}) - \hat{v}(S_{k+1}) - (v(S_k) - \hat{v}(S_k))| < \epsilon,$$

where v *and* $\hat{v}$ *are defined by*

$$(40.3)\quad v(S) = u_S(\textstyle\int_S \underset{\sim}{a}) \text{ and } \hat{v}_S = \hat{u}_S(\textstyle\int_S \underset{\sim}{a}) \text{ for all } S.$$

Proof. Set $A_k = v(S_{k+1}) - \hat{v}(S_{k+1}) - (v(S_k) - \hat{v}(S_k))$. We start out by fixing attention on a single k. To simplify the notation by eliminating the need for a large number of subscripts, set $S_{k+1} = S$, $u_{S_{k+1}} = w$, $\int_{S_{k+1}} \underset{\sim}{a} = b$, and let $v(S) = w(b)$ be attained at $\underset{\sim}{y}$. Similarly, set $S_k = S_o$,

$u_{S_k} = w_o$, $\int_{S_k} \underset{\sim}{a} = b_o$, and let $v(S_o) = w_o(b_o)$ be attained at $\underset{\sim}{y}_o$. Adopt a similar notation for $\hat{v}$, $\hat{u}$, etc. Also set $V = S \backslash S_o$.

For $0 \leqq \theta \leqq 1$, set

$$g(\theta) = w_o(\theta \int_{S_o} \underset{\sim}{y} + (1 - \theta) b_o).$$

The function g is continuous on the closed unit interval [0,1], and is differentiable in the interior (0,1). The continuity follows from Proposition 37.13; to prove the differentiability, let

$$c_\theta = \theta \int_{S_o} \underset{\sim}{y} + (1 - \theta) b_o.$$

Then since $b_o = \int_{S_o} \underset{\sim}{a} > 0$, it follows that $c_\theta > 0$ for all $\theta \in (0,1)$. Applying Corollary 38.13, we deduce that $w_o^j(x)$ exists and is continuous at $x = c_\theta$ for each j and each θ in (0,1). We conclude that $g'(\theta)$ exists for all $\theta \in (0,1)$, and

$$g'(\theta) = w'(c_\theta) \cdot \Delta,$$

where $\Delta \in E^n$ is defined by

$$\begin{aligned}\Delta &= \int_{S_o} \underset{\sim}{y} - b_o \\ &= [\int_S \underset{\sim}{y} - \int_V \underset{\sim}{y}] - [\int_S \underset{\sim}{a} - \int_V \underset{\sim}{a}] \\ &= \int_V (\underset{\sim}{a} - \underset{\sim}{y}).\end{aligned}$$

Since g is continuous on [0,1] and differentiable in (0,1), the mean value theorem applies, and we deduce that $g(1) - g(0) = g'(\theta)$ for an appropriate θ. Setting $c = c_\theta$ for that θ, we obtain

$$w_o(\int_{S_o} \underset{\sim}{y}) - w_o(b_o) = w_o'(c)\cdot\Delta. \tag{40.4}$$

Now w(b) is attained at $\underset{\sim}{y}$; hence by Proposition 38.5,

$$u(x, s) - u(\underset{\sim}{y}(s), s) \leqq w'(b)\cdot(x - \underset{\sim}{y}(s)) \tag{40.5}$$

for almost all $s \in S$ and almost all $x \in \Omega$; in particular this is true for almost all $s \in S_o$. Since trivially we have $\int_{S_o} \underset{\sim}{y} = \int_{S_o} \underset{\sim}{y}$, it follows from Proposition 36.4 that $w_o(\int_{S_o} \underset{\sim}{y})$ is attained at $\underset{\sim}{y}|S_o$; that is, we have

$$w_o(\int_{S_o} \underset{\sim}{y}) = \int_{S_o} u(\underset{\sim}{y}). \tag{40.6}$$

By applying Proposition 38.5 to S_o we obtain

$$u(x, s) - u(\underset{\sim}{y}(s), s) \leqq w_o'\left(\int_{S_o} \underset{\sim}{y}\right) \cdot (x - \underset{\sim}{y}(s))$$

whenever $s \in S_o$ and $\int_{S_o} \underset{\sim}{y} > 0$. From this and (40.5) it follows easily that if $\int_{S_o} \underset{\sim}{y} > 0$, then

$$(40.7) \qquad w(b) = w_o'\left(\int_{S_o} \underset{\sim}{y}\right)$$

(since for each j there must be an s in S_o with $\underset{\sim}{y}^j(s) > 0$). Formula (40.7) is needed for later reference; at the moment we need only (40.6), which, together with (40.4) yields

$$w_o(b_o) = - w_o'(c) \cdot \Delta + \int_{S_o} u(\underset{\sim}{y}).$$

Since $w(b) = \int_S u(\underset{\sim}{y})$ by definition, it follows that

$$(40.8) \qquad w(b) - w_o(b_o) = \int_V u(\underset{\sim}{y}) + w_o'(c) \cdot \Delta.$$

If we go through the above argument for $\hat{u}$, defining $\hat{c}$ and $\hat{\Delta}$ in the appropriate fashion, i.e., by setting

$$\hat{z} = \hat{\theta}\int_{S_o} \hat{y} + (1 - \hat{\theta})b_o$$

for an appropriate $\hat{\upsilon} \in [0,1]$ and

$$\hat{\Delta} = \int_V (\underset{\sim}{a} - \hat{y}),$$

then it follows that

$$\hat{w}(b) - \hat{w}_o(b_o) = \int_V \hat{u}(\hat{y}) + \hat{w}_o'(\hat{c})\cdot\hat{\Delta}.$$

We note also that

(40.10) $u(\hat{y}(s), s) - \hat{w}'(b)\cdot y(s) \leqq \hat{u}(\hat{y}(s), s) - \hat{w}'(b)\cdot\hat{y}(s)$

for all $s \in S$; this follows from Proposition 38.5. Similarly

$$u(\hat{y}(s), s) - w'(b)\cdot\hat{y}(s) \leqq u(y(s), s) - w'(b)\cdot y(s).$$

Now let

$$I_1 = \int_V [u(\underset{\sim}{y}) - \hat{u}(\underset{\sim}{y})]$$
$$\hat{I}_1 = \int_V [\hat{u}(\hat{\underset{\sim}{y}}) - u(\hat{\underset{\sim}{y}})]$$
$$I_2 = [w'(b) - \hat{w}'(b)]\cdot\Delta$$
$$\hat{I}_2 = [\hat{w}'(b) - w'(b)]\cdot\hat{\Delta}$$
$$I_3 = [w_o'(c) - w'(b)]\cdot\Delta$$
$$\hat{I}_3 = [\hat{w}_o'(\hat{c}) - \hat{w}'(b)]\cdot\hat{\Delta}$$
$$J_1 = [\hat{w}_o'(\hat{c}) - w_o'(\hat{c})]\cdot\hat{\Delta}$$
$$J_2 = [w_o'(\hat{c}) - w'(b)]\cdot\hat{\Delta}.$$

Note that

$$\hat{I}_3 = J_1 + J_2 - \hat{I}_2.$$

Then we have

$$
\begin{aligned}
(40.11)\qquad A_k &= w(b) - w_o(b_o) - (\hat{w}(b) - \hat{w}_o(b_o)) \\
&= \int_V u(\underset{\sim}{y}) + \hat{w}'(b)\cdot\Delta + (w_o'(c) - w'(b))\cdot\Delta \\
&\quad - \int_V [\hat{u}(\hat{\underset{\sim}{y}}) - \hat{w}'(b)\cdot\hat{\underset{\sim}{y}} + w'(b)\cdot\underset{\sim}{a}] - (\hat{w}_o'(\hat{c}) - \hat{w}'(b))\cdot\hat{\Delta} \\
&\leqq \int_V u(\underset{\sim}{y}) + w'(b)\cdot\Delta + I_3 \\
&\quad - \int_V [\hat{u}(\underset{\sim}{y}) - \hat{w}'(b)\cdot\underset{\sim}{y} + \hat{w}'(b)\cdot\underset{\sim}{a}] - \hat{I}_3 \\
&= I_1 + I_2 + I_3 - \hat{I}_3,
\end{aligned}
$$

where the inequality follows from (40.10). Similarly

$$(40.12)\quad -A_k \leqq \hat{I}_1 + \hat{I}_2 + \hat{I}_3 - I_3 = \hat{I}_1 - I_3 + J_1 + J_2.$$

The proof of Lemma 40.1 will be completed by estimating the quantities I_i, $\hat{I}_i$, and J_i. These quantities can be made sufficiently small (in absolute value) to prove (40.2) if the δ appearing in the statement of the lemma is appropriately chosen.

It will be useful to make the following definition: a quantity is <u>uniformly small when</u> δ <u>is chosen sufficiently small</u> if for every $\epsilon_1 > 0$, it is less than ϵ_1 in absolute value, for appropriate choice of δ, uniformly in $\hat{u}$, S_0, and S (i.e., uniformly in $\hat{u}$, in the choice of the chain $\emptyset \subset S_1 \subset S_2 \subset \ldots$, and in the choice of a particular link in this chain).

Let $\alpha = \Sigma\int \underset{\sim}{a}$. By Lemma 37.8, there is an integrable $\underset{\sim}{\eta}$, depending on u and ϵ only, such that if δ is sufficiently small then $\underset{\sim}{y}(s) \leqq \underset{\sim}{\eta}(s)e$ for almost all $s \in S$. Choose $\underset{\sim}{\eta}$ so that also $\underset{\sim}{a}(s) \leqq \underset{\sim}{\eta}(s)$. From this it follows that

$$\|\Delta\| \leqq \int_V \underset{\sim}{\eta}, \quad \|\hat{\Delta}\| \leqq \int_V \underset{\sim}{\eta}. \tag{40.13}$$

Next, let D be the exceptional set in the definition of δ-approximation, i.e., the set in which we do not necessarily have $\|u_s - \hat{u}_s\| \leqq \delta$. Let $\underset{\sim}{\zeta}$ be an integrable real function such that $u(x, s) \leqq \underset{\sim}{\zeta}(s) + \Sigma x$ for all s in I and all x in Ω; such a $\underset{\sim}{\zeta}$ exists by Lemma 37.9. Then

$$(40.14)\qquad |I_1| \leq \int_V |u(\underset{\sim}{y}) - \hat{u}(\underset{\sim}{y})| = \int_{V\setminus D} + \int_{V\cap D}$$

$$\leq \delta\int_{V\setminus D}(1 + \Sigma\underset{\sim}{y}) + \int_{V\cap D}(\underset{\sim}{\zeta} + \Sigma\underset{\sim}{y} + \sqrt{\Sigma\underset{\sim}{y}})$$

$$\leq \delta\int_V (1 + n\underset{\sim}{\eta}) + \int_{V\cap D}(\underset{\sim}{\zeta} + 2n\underset{\sim}{\eta}).$$

Similarly,

$$(40.15)\qquad |\hat{I}_1| \leq \delta\int_{V\setminus D}(1 + n\underset{\sim}{\eta}) + \int_{V\cap D}(\underset{\sim}{\zeta} + 2n\eta).$$

Next, we must estimate the terms that multiply Δ and $\hat{\Delta}$ in the expressions for I_2, $\hat{I}_2$, I_3, $\hat{I}_3$, J_1 and J_2. For this purpose note first that by (40.7), if $\int_{S_o}\underset{\sim}{y} > 0$, then

$$(40.16)\qquad I_3 = (w_o'(c) - w_o'(\int_{S_o}\underset{\sim}{y}))\cdot\Delta,$$

$$(40.17)\qquad J_2 = (w_o'(\hat{c}) - w_o'(\int_{S_o}\underset{\sim}{y}))\cdot\hat{\Delta};$$

and similarly if $\int_{S_o}\hat{\underset{\sim}{y}} > 0$, then

$$(40.18)\qquad \hat{I}_3 = (\hat{w}_o'(\hat{c}) - \hat{w}_o'(\int_{S_o}\hat{\underset{\sim}{y}}))\cdot\hat{\Delta}.$$

Now

$$c - \int_{S_o} \chi = \theta\int_{S_o} \chi + (1 - \theta)b_o - \int_{S_o} \chi$$

$$= (1 - \theta)(b_o - \int_{S_o} \chi) = -(1 - \theta)\Delta.$$

Hence

$$\|c - \int_{S_o} \chi\| \leqq \|\Delta\|,$$

and similarly

$$\|\hat{c} - \int_{S_o} \hat{\chi}\| \leqq \|\hat{\Delta}\|.$$

Combining this with (40.13) and with the fact that $\mu(V) \leqq \delta$, we obtain that if δ is chosen sufficiently small, then

(40.19) $\|c - \int_{S_o} \chi\|$ and $\|\hat{c} - \int_{S_o} \hat{\chi}\|$ are uniformly small.

Furthermore, we have

$$\hat{c} - \int_{S_o} \chi = \hat{c} - c + c - \int_{S_o} \chi = \theta(\hat{\Delta} - \Delta) + c - \int_{S_o} \chi.$$

Hence

$$\|\hat{c} - \int_{S_o} \underset{\sim}{y}\| \leqq \|\hat{\Delta}\| + \|\Delta\| + \|c - \int_{S_o} \underset{\sim}{y}\|,$$

and combining this with (40.13) and (40.19) we obtain as above that if δ is chosen sufficiently small, then

(40.20) $\|\hat{c} - \int_{S_o} \underset{\sim}{y}\|$ is uniformly small.

We now make use of the assumption that $\int_{S_1} \underset{\sim}{a} \geqq \epsilon e$. If we recall that $S_o = S_k \supset S_1$ and that $b_o = \int_{S_o} \underset{\sim}{a}$, then from (40.13) and $\mu(V) \leqq \delta$ it follows that when δ is sufficiently small, $\int_{S_o} \underset{\sim}{y} \geqq \frac{1}{2}\epsilon e$ and $\int_{S_o} \hat{\underset{\sim}{y}} \geqq \frac{1}{2}\epsilon e$. Hence the vectors b, c, $\hat{c}$, $\int_{S_o} \underset{\sim}{y}$ and $\int_{S_o} \hat{\underset{\sim}{y}}$ are all in $A(\frac{1}{2}\epsilon, \alpha)$. So we may apply Propositions 38.7 and 38.14, and formulas (40.16) through (40.20), and deduce that if δ is chosen sufficiently small, then the terms multiplying Δ and $\hat{\Delta}$ in the definitions of I_2, $\hat{I}_2$, I_3, $\hat{I}_3$, J_1 and J_2 will be uniformly small. Taking into account formulas (40.11) through (40.15), we deduce that for any given $\epsilon_1 > 0$, if δ is chosen sufficiently small, then

(40.21) $|A_k| \leqq \delta\int_V (1 + n\underset{\sim}{\eta}) + \int_{V\cap D} (\underset{\sim}{\zeta} + 2n\underset{\sim}{\eta}) + \epsilon_1\int_V \underset{\sim}{\eta};$

here δ, $\underset{\sim}{\eta}$ and $\underset{\sim}{\zeta}$ depend on u, $\underset{\sim}{a}$, ϵ, and ϵ_1 only, and not on the choice of $\hat{u}$ or the S_k (providing, of course, that $\hat{u}$ and the S_k satisfy the conditions of the lemma). Writing V_k for V and recalling that $V_k = V = S\backslash S_o = S_{k+1}\backslash S_k$, we deduce that the V_k are mutually disjoint, and their union is included in I; similarly, the $V_k \cap D$ are mutually disjoint, and their union is included in D. Hence from (40.21) we get

$$\Sigma_{k=1}^{m} |A_k| \leqq \delta\int(1 + n\underset{\sim}{\eta}) + \int_D(\underset{\sim}{\zeta} + 2n\underset{\sim}{\eta}) + \epsilon_1\int\underset{\sim}{\eta}.$$

Using the integrability of $\underset{\sim}{\eta}$ and $\underset{\sim}{\zeta}$ and the fact that $\mu(D) \leqq \delta$, we deduce that if δ and ϵ_1 are chosen small enough, then

$$\Sigma_{k=1}^{m} |A_k| \leqq \epsilon.$$

This completes the proof of Lemma 40.1.

PROPOSITION 40.22. *Let* $u \in \mathcal{U}_1$, *and let* $\underset{\sim}{a}$ *be* μ-*integrable. Then for each* $\epsilon > 0$ *there is a* $\delta > 0$ *such that if* $\hat{u} \in \mathcal{U}_1$ *is a* δ-*approximation to* u, *and* $\int_{\underset{\sim}{S}} a \leqq \delta e$, *then* $\hat{v}(S) < \epsilon$, *where* $\hat{v}$ *is defined by* (40.3).

<u>Proof</u>. Clearly $u_I(0) = 0$. By Proposition 37.13, u_I is continuous on Ω. Hence for δ sufficiently small, $\int_S \underset{\sim}{a} \leq \delta e$ yields

$$u_I(\int_S \underset{\sim}{a}) < \frac{1}{2}\varepsilon. \tag{40.23}$$

On the other hand, if in Proposition 37.11 we substitute $\frac{1}{4}\varepsilon$ for ε, then it follows that for δ sufficiently small, we have

$$|u_I(\int_S \underset{\sim}{a}) - \hat{u}_I(\int_S \underset{\sim}{a})| < \frac{1}{4}\varepsilon(1 + \Sigma\int_S \underset{\sim}{a}) \leq \frac{1}{4}\varepsilon(1 + 1) = \frac{1}{2}\varepsilon.$$

Combining this with (40.23), we get

$$\hat{v}(S) = \hat{u}_S(\int_S \underset{\sim}{a}) \leq \hat{u}_I(\int_S \underset{\sim}{a}) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

This completes the proof of Lemma 40.22.

<u>PROPOSITION 40.24</u>. <u>Let</u> $u \in \mathcal{U}_1$, <u>let</u> $\underset{\sim}{a}$ <u>be</u> μ-<u>integrable, and assume</u>

(40.25) <u>for all</u> $s \in I$, <u>either</u> $\underset{\sim}{a}(s) > 0$ <u>or</u> $\underset{\sim}{a}(s) = 0$.

<u>Then for each</u> $\varepsilon > 0$ <u>there is a</u> $\delta > 0$ <u>such that</u> <u>if</u> $\hat{u} \in \mathcal{U}_1$ <u>is a</u> δ-<u>approximation to</u> u, <u>then</u>

$$\|v - \hat{v}\| < \varepsilon,$$

<u>where</u> v <u>and</u> $\hat{v}$ <u>are defined by</u> (40.3).

<u>Remark</u>. Condition (40.25) says that for each s, either all coordinates of $\underset{\sim}{a}(s)$ are positive or all vanish. This condition is implied both by (31.4) and by $n = 1$; we will use it to state (and prove) below a common generalization of Theorem G and Proposition 33.2.

<u>Proof</u>. Let δ_1 correspond to $\frac{1}{4}\varepsilon$ in accordance with Lemma 40.22. Choose γ so that* $0 < \gamma < \int \underset{\sim}{a}^1$ and so that $\int_S \underset{\sim}{a}^1 \leqq \gamma$ implies $\int_S \underset{\sim}{a} < \delta_1 e$; this is possible because of (40.25). Choose $\varepsilon_1 > 0$ so that $\int_S \underset{\sim}{a}^1 \geqq \gamma$ implies $\int_S \underset{\sim}{a} > \varepsilon_1 e$; this, again, is possible because of (40.25). Let

$$\varepsilon_2 = \min\left(\varepsilon_1, \tfrac{1}{2}\varepsilon\right),$$

and choose δ_2 to correspond to ε_2 in accordance with Lemma 40.1. Let $\delta = \min(\delta_1, \delta_2)$.

Let $w = v - \hat{v}$, and let

$$\emptyset = S_0 \subset S_1 \subset \dots \subset S_m \subset S_{m+1} = I$$

be a chain. It is always possible to insert finitely

---

*If $\int \underset{\sim}{a}^1 = 0$, then from (40.25) if follows that $\int \underset{\sim}{a} = 0$, and the whole problem becomes trivial.

many additional sets $S_{01}, S_{02}, \ldots, S_{11}, S_{12}, \ldots, \ldots,$ $S_{m1}, S_{m2}, \ldots$ into the chain so that $S_0 \subset S_{01} \subset S_{11} \subset \ldots \subset S_1 \subset S_{11} \subset S_{12} \subset \ldots \subset \ldots \subset S_m \subset S_{m1} \subset S_{m2} \subset \ldots \subset S_{m+1}$ and the measure of the difference between two neighboring sets is $< \delta$; that is, if we relabel the new sequence $U_0, \ldots, U_{p+1} = I$, then $\mu(U_{k+1} \setminus U_k) < \delta$ for all k. Furthermore, by Lyapunov's theorem in one dimension we may suppose w.l.o.g. that for one of the U_k, say for U_q, we have $\int_{U_q} \tilde{a}^1 = \gamma$. Then

$$\sum_{k=0}^{m} |w(S_{k+1}) - w(S_k)| \leqq \sum_{k=0}^{p} |w(U_{k+1}) - w(U_k)| = \sum_{k=0}^{q-1} + \sum_{k=q}^{p} .$$

Lemma 40.1 and the fact that $\hat{u}$ is a δ-approximation--hence a fortiori a δ_2-approximation--to u yield

$$\sum_{k=q}^{p} \leqq \epsilon_2 \leqq \frac{1}{2}\epsilon .$$

Furthermore, since $\hat{u}$ is a δ-approximation to u, it is a fortiori a δ_1-approximation. Hence by the monotonicity of v and $\hat{v}$ and Lemma 40.24, we have

$$\sum_{k=0}^{q-1} |w(U_{k+1}) - w(U_k)|$$

$$= \sum_{k=0}^{q-1} |v(U_{k+1}) - v(U_k) - (\hat{v}(U_{k+1}) - \hat{v}(U_k))|$$

$$\leq \sum_{k=0}^{q-1} |v(U_{k+1}) - v(U_k)| + \sum_{k=0}^{q-1} |\hat{v}(U_{k+1}) - \hat{v}(U_k)|$$

$$= \sum_{k=0}^{q-1} (v(U_{k+1}) - v(U_k)) + \sum_{k=0}^{q-1} (\hat{v}(U_{k+1}) - \hat{v}(U_k))$$

$$= v(U_q) + \hat{v}(U_q) < \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon = \tfrac{1}{2}\varepsilon ,$$

because $\int_{U_q} a < \delta_1 e$ (note that u is a 0-approximation, hence trivially a δ_1-approximation, to itself).

We conclude that

$$\sum_{k=0}^{m} |w(S_{k+1}) - w(S_k)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon ,$$

and it follows that $\|v - \hat{v}\| = \|w\| < \varepsilon$. This completes the proof of Proposition 40.24.

<u>PROPOSITION 40.26</u>. <u>Theorem</u> G <u>holds if</u> (31.4) <u>is replaced by</u> (40.25).

<u>Proof</u>. Recall that H is the set of all superadditive set-functions in pNA that are homogeneous of degree 1. By Proposition 35.6, for every δ there is a δ-approximation $\hat{u}$ to u that is of finite type. If $\hat{v}$ corresponds to the given

$\underset{\sim}{a}$ and to this $\hat{u}$ (in accordance with (40.3)), then by Proposition 40.24, for given ϵ we will have $\|\hat{v} - v\| < \epsilon$ when δ is sufficiently small. But by Lemma 39.16, $\hat{v} \in H$; thus v can be approximated in variation by members of H, i.e., it is in the closure of H. But H is closed (Proposition 27.12), and so we have proved that $v \in H$. Proposition 40.26 now follows from Theorem F.

Theorem G and Proposition 33.2 both follow immediately from Proposition 40.26.

41. THE ASYMPTOTIC VALUE OF A MARKET

Throughout this section, pNAD will denote the closure of pNA + DIAG (see Section 19 for the definition of DIAG).

PROPOSITION 41.1. There is a value φ on pNAD that is continuous in the variation norm and enjoys the diagonal property; furthermore, there is only one such value. Finally,

$$\text{pNAD} \subset \text{ASYMP},$$

and the value φ coincides with the asymptotic value on pNAD.

Proof. We have pNA $\subset$ ASYMP (Proposition 18.6), DIAG $\subset$ ASYMP (Proposition 19.7), and ASYMP is a closed linear subspace of BV (Proposition 18.4). Hence pNAD = $\overline{\text{pNA + DIAG}} \subset$ ASYMP, and a fortiori pNAD $\subset$ ASYMP.

It remains to prove that there is at most one value on pNAD that is continuous and enjoys the diagonal property. Indeed, if φ is a value on pNAD with these properties, then φ is determined on pNA by the uniqueness of the value on pNA (Proposition 7.11), and on DIAG it must vanish identically by the diagonal property (Section 19). Hence it is determined on pNA + DIAG, and so by continuity on its closure, namely on pNAD. This completes the proof of Proposition 41.1.

Convention. For the remainder of this section, φ will denote the unique value on pNAD provided by Proposition 41.1.

We are now ready to state the main result of this section, which, together with Proposition 41.1, immediately implies Proposition 31.7.

PROPOSITION 41.2. Let $\underset{\sim}{a}$ be μ-integrable, let $u \in \mathcal{U}_1$, and let v be given by (30.1). Then v is well-defined and is in pNAD, and the core of v consists of the single point φv.

The proof of Proposition 41.2 will proceed in two stages. First we shall prove a generalization of Theorem F (Proposition 41.28). In this generalization the hypothesis $v \in$ pNA is replaced by*

$$v \in \text{pNAD} \cap \text{pNA}'. \tag{41.3}$$

Since $v \in$ pNA', the extension v^* is defined,** and hence the homogeneity condition in Theorem F makes sense. In the conclusion, we are no longer justified in speaking of "the" value; however, the conclusion remains true if one refers to the unique value φ on pNAD provided by Proposition 41.1.

*Recall that pNA' is the closure of pNA in the supremum norm; see Section 22.

**See Proposition 22.10.

In the second stage of the proof, we shall prove that the v of Proposition 41.2 satisfies (41.3), and that it is also superadditive and homogeneous of degree 1. The proof of Proposition 41.2--and hence of Proposition 31.7--is then easily completed.

In the next few lemmas we shall make free use of the notations and terminology of Part III, in particular of Sections 22 and 23. We begin with a generalization of Lemma 22.1.

LEMMA 41.4. *Let* ξ *be a finite-dimensional vector of measures in* NA. *Let* $g_1, \ldots, g_m$ *be ideal set functions with*

$$g_1 \leq \cdots \leq g_m.$$

Then there are sets $T_1, \ldots, T_m$ *in* $\mathcal{C}$ *with*

$$T_1 \subset \ldots \subset T_m$$

such that for all i,

$$\xi(T_i) = \int g_i d\xi.$$

Proof. The proof is exactly analogous to that of Lemma 22.1.

LEMMA 41.5. Let $v \in BV \cap pNA'$. Then $v^* \in IBV$, and $\|v^*\| = \|v\|$.

Remark. When we write $\| \ \|$ we are, of course, referring to the variation norm. The supremum norm is denoted by $\| \ \|'$, and the equation $\|v^*\|' = \|v\|'$ has already been established (see (22.8)).

Proof. Let Ω be a chain

$$0 = g_0 \leq g_1 \leq \cdots \leq g_m \leq g_{m+1} = \chi_I$$

of ideal set functions. For a given ϵ, let $\xi_1, \ldots, \xi_m$ be vectors of measures in NA, and $\delta_1, \ldots, \delta_m$ positive numbers, such that for all i,

$$\|\int (f - g_i) d\xi_i\| \leq \delta_i \Rightarrow |v^*(f) - v^*(g_i)| < \epsilon;$$

the existence of such ξ_i and δ_i follows from the continuity of v^* in the NA-topology (see (22.6)). Let ξ be the vector $(\xi_1, \ldots, \xi_m)$, and define $T_1, \ldots, T_m$ in accordance with Lemma 41.5. Then $\int (\chi_{T_i} - g_i) d\xi = 0$ for all i, and hence $\int (\chi_{T_i} - g_i) d\xi_i = 0 \leq \delta_i$ for all i. Hence

$$|v(T_i) - v^*(g_i)| = |v^*(\chi_{T_i}) - v^*(g_i)| < \epsilon.$$

Setting $T_0 = \emptyset$ and $T_{m+1} = I$, we deduce

$$\|v^*\|_\Omega = \Sigma_{i=0}^m |v^*(g_{i+1}) - v^*(g_i)|$$

$$\leqq \Sigma_{i=0}^m |v(T_{i+1}) - v(T_i)| + 2(m+1)\epsilon \leqq \|v\| + 2(m+1)\epsilon .$$

Letting $\epsilon \to 0$, we deduce $\|v^*\|_\Omega \leqq \|v\|$, and hence

$$\|v^*\| = \sup_\Omega \|v^*\|_\Omega \leqq \|v\|.$$

Since the inequality $\|v\| \leqq \|v^*\|$ is obvious, the proof of Lemma 41.5 is complete.

LEMMA 41.6. *Let* $T_1, \ldots, T_m$ *be disjoint measurable subsets of* $(0, 1)$. *With each* t *in each* T_i, *let there be associated a family* $\mathscr{K}^t$ *of closed intervals* in* $(0, 1)$, *one of whose endpoints is* t; *assume moreover, that each* $\mathscr{K}^t$ *contains arbitrarily short intervals. Let*

$$\mathscr{K}_i = \cup_{t \in T_i} \mathscr{K}^t, \quad \text{and} \quad \mathscr{K} = \cup_{i=1}^m \mathscr{K}_i.$$

Then for each $\epsilon > 0$, *there is a finite family* $\mathscr{S}$ *of mutually disjoint intervals in* $\mathscr{K}$, *such that if* S_i *is the union of the intervals in* $\mathscr{S} \cap \mathscr{K}_i$, *then*

*All intervals in this lemma and its proof are understood to have positive length.

$$\lambda(S_i + T_i) < \epsilon,$$

<u>where</u> λ <u>is Lebesgue measure and</u> "+" <u>denotes the symmetric difference</u>.

<u>Proof</u>. Let us call T_i <u>pure left</u> if for each t in T_i, $\mathscr{K}^t$ contains arbitrarily short intervals whose left endpoint is t. Define <u>pure right</u> analogously, and call T_i <u>pure</u> if it is either pure left or pure right (or both).

First we prove the lemma in the case in which each T_i is pure, proceeding by induction on m. The case in which m = 1 and T_i is pure left is proved in [T, §11.41, Lemmas 1 and 2, pp. 356-357]; the pure right case, of course, follows from the pure left case by symmetry arguments.

Now assume that the lemma has been proved for m - 1; and let $T_1, \ldots, T_m$, and the sets $\mathscr{K}^t$, $\mathscr{K}_i$ and $\mathscr{K}$ satisfy the hypotheses of the lemma. Let $\mathscr{K}_* = \mathscr{K}_1 \cup \ldots \cup \mathscr{K}_{m-1}$. Applying the induction hypothesis (for ϵ/m instead of ϵ), we obtain a family $\mathscr{S}_*$ of mutually disjoint intervals in $\mathscr{K}_*$, such that for i = 1, ..., m - 1, if S_i is the union of the intervals in $\mathscr{S}_* \cap \mathscr{K}_i$, then

$$(41.7) \qquad \lambda(S_i + T_i) < \epsilon/m.$$

Let $T_* = T_1 \cup \ldots \cup T_{m-1}$, and $S_* = S_1 \cup \ldots \cup S_{m-1}$. Note that

$$S_* + T_* \subset (T_1 + S_1) \cup \ldots \cup (T_{m-1} + S_{m-1}),$$

and hence

$$\lambda(S_* + T_*) \leq \frac{m-1}{m} \varepsilon < \varepsilon.$$

Next, since T_m is pure, we may assume w.l.o.g. that it is pure left. Let $T_{**} = T_m/S_*$, and for $t \in T_{**}$, let $\mathscr{V}^t_{**}$ be the set of all intervals in $\mathscr{V}^t_{**}$ whose left-hand end-point is t and which do not intersect S_*. $\mathscr{V}^t$ is nonempty and contains arbitrarily small intervals, because S_* is closed and $\mathscr{V}^t$ contains arbitrarily small intervals whose left-hand end point is t. Let $\mathscr{V}_{**} = \cup_{t \in T_m} \mathscr{V}^t_{**}$. Apply the case $m = 1$ (with ε/m instead of ε) to T_{**} and $\mathscr{V}_{**}$, obtaining a finite family $\mathscr{S}_m$ of disjoint intervals in $\mathscr{V}_{**}$ such that

$$\lambda(S_m + T_{**}) < \varepsilon/m,$$

where S_m is the union of the intervals in $\mathscr{S}_m$. Let $\mathscr{S} = \mathscr{S}_* \cup \mathscr{S}_m$. Since $S_* \cap S_m = \emptyset$, the members of $\mathscr{S}$ are disjoint. Furthermore, it may be verified that

$$S_m + T_m \subset (S_m + T_{**}) \cup (S_* \backslash T_*);$$

hence

$$\lambda(S_m + T_m) < \lambda(S_m + T_{**}) + \lambda(S_* + T_*) < \frac{\varepsilon}{m} + \frac{m-1}{m}\varepsilon = \varepsilon .$$

This, together with (41.7), completes the proof of the lemma in the case in which all the T_i are pure.

In the general case, let T_i^L, for each i, be the set of all t in T_i for which $\mathscr{V}_t$ contains arbitrarily small intervals whose left-hand end-point is t; let $T_i^R = T_i \backslash T_i^L$. Let $\mathscr{V}_i^L = \bigcup_{t \in T_i^L} \mathscr{V}^t$, and $\mathscr{V}_i^R = \bigcup_{t \in T_i^R} \mathscr{V}^t$. Then T_i^L is pure left and T_i^R is pure right, so we may apply the case just proved to the system consisting of $T_1^L, \ldots, T_m^L, T_1^R, \ldots, T_m^R$, and the $\mathscr{V}^t$. If we use $\frac{1}{2}\varepsilon$ instead of ε, this yields a finite family $\mathscr{S}$ of mutually disjoint intervals in $\mathscr{V}$, such that if S_i^L and S_i^R is the union of the intervals in $\mathscr{S} \cap \mathscr{V}_L^i$ and $\mathscr{S} \cap \mathscr{V}_R^i$ respectively, then

$$\lambda(S_i^L + T_i^L) < \frac{1}{2}\varepsilon \text{ and } \lambda(S_i^R + T_i^R) < \frac{1}{2}\varepsilon .$$

Since

$$S_i + T_i \subset (S_i^L + T_i^L) \cup (S_i^R + T_i^R),$$

it follows that

$$\lambda(S_i + T_i) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

and the proof of the lemma is complete.

<u>LEMMA 41.8</u>. <u>Let</u> f <u>be a nonnegative extended real-valued* function on</u> $(0, 1)$. <u>For each positive integer</u> k <u>and each</u> i <u>with</u> $0 \leqq i \leqq k^2$, <u>define</u>

$$T_{ik} = \begin{cases} \{t : i/k \leqq f(t) < (i+1)/k\} & \text{for } i < k^2 \\ \{t : k \leqq f(t)\} & \text{for } i = k^2. \end{cases}$$

<u>Then</u>

$$\Sigma_{i=0}^{k^2} \frac{i}{k} \lambda(T_{ik}) \to \int_0^1 f(t)dt$$

<u>as</u> $k \to \infty$.

<u>Proof</u>. This follows easily from any of the standard definitions of the Lebesgue integral.

Let $v^* \in IBV$. For each $\delta > 0$, define the δ-<u>norm</u> $\|v^*\|_\delta$ to be

$$\sup \Sigma_{i=0}^{m} |v^*(g_{i+1}) - v^*(g_i)|,$$

where the sup is over all chains

*I.e., f may take the value $+\infty$ as well as finite nonnegative values.

$$0 = g_0 \leqq g_1 \leqq \cdots \leqq g_m \leqq g_{m+1} = \chi_I$$

such that for all g_i and all s, $s' \in I$, we have

(41.9) $$|g_i(s) - g_i(s')| < \delta.$$

The restriction (41.9) means that the g_i are "close" to the diagonal; indeed, if $g_i(s) = g_i(s')$ for all s, s' in I, then g_i is of the form $t\chi_I$, and so is on the diagonal. Thus $\|v^*\|_\delta$ is the sup of the variation of v^* over chains which always remain in a δ-neighborhood of the diagonal. Note that $\|v^*\|_\delta \leqq \|v^*\|$; and so if $v \in BV \cap pNA'$, then by Lemma 41.5, it follows that

(41.10) $$\|v^*\|_\delta \leqq \|v\|.$$

The next lemma generalizes the hypothesis as well as sharpens the conclusion of Lemma 23.1.

<u>LEMMA 41.11</u>. <u>Let</u> $v \in BV \cap pNA'$, <u>and let</u> $S \in \mathcal{C}$. <u>Then</u> $|\partial v^*(t)|^+$ <u>is integrable over</u> $[0,1]$, <u>and for all</u> $\delta > 0$,

$$\int_0^1 |\partial v^*(t)|^+ dt \leqq \|v^*\|_\delta.$$

<u>Proof</u>. Fix $\delta > 0$. Let k be a positive integer. Define a partition $\{T_0, T_1, \ldots, T_{k^2}\}$ of $(0, 1)$ by

$$(41.12)\quad T_i = \begin{cases} \{t : i/k \leq |\partial v^*(t)|^+ < (i+1)/k\} & \text{for } i < k^2 \\ \{t : k \leq |\partial v^*(t)|^+\} & \text{for } i = k^2. \end{cases}$$

For each $t \in (0,1)$ there are numbers τ, arbitrarily small in absolute value, such that

$$(41.13)\quad \begin{gathered} \tau \neq 0,\ |\tau| < \delta,\ |t+\tau| \in (0,1), \quad \text{and} \\ |\partial v^*(t)|^+ - \left|\frac{v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I)}{\tau}\right| < \frac{1}{k}. \end{gathered}$$

With each t and τ satisfying (41.13), associate the interval whose endpoints are t and $t + \tau$; let $\mathscr{V}^t$ be the family of intervals so defined, Now apply Lemma 41.6 to the system defined by $\{T_0, T_1, \ldots, T_{k^2}\}$, and the families $\mathscr{V}^t$. This yields a family $\mathscr{S}$ of intervals satisfying the conclusions of that lemma (for given ε). Denote the intervals of $\mathscr{S}$ by $U_1, \ldots, U_p$, where for all h, the right end-point of U_h is left of the left end-point of U_{h+1}; this is possible because the U_h are disjoint. Each U_h has end-points t and $t + \tau$ satisfying (41.13); denote them by t_h and $t_h + \tau_h$ respectively. Now construct a chain Ω of ideal sets

$$0 = g_0 \leq g_1 \leq \cdots \leq g_{2p+1} = \chi_I$$

by letting

$$g_{2h-1} = t_h\chi_I \quad \text{and} \quad g_{2h} = t_h\chi_I + \tau_h\chi_S$$

if $\tau > 0$, and

$$g_{2h-1} = t_h\chi_I + \tau_h\chi_S \quad \text{and} \quad g_{2h} = t_h\chi_I$$

if $\tau < 0$. Note that since $|\tau_h| < \delta$ for all h, the chain Ω satisfies the condition (41.9), and hence $\|v^*\|_\delta \geqq \|v^*\|_\Omega$. To evaluate $\|v^*\|_\Omega$, for each i relabel the intervals constituting $\mathcal{S} \cap \mathcal{X}_i$ by $U_{i1}, \ldots, U_{iq}$ (where q depends on i); these are some of the U_h, and when i varies, we get all of the U_h. If $U_{ij} = U_h$, let $t_{ij} = t_h$ and $\tau_{ij} = \tau_h$. Then $t_{ij} \in T_i$, and so by Lemma 41.6 and Formula (41.12), we have

$$\begin{aligned}
\|v^*\|_\delta \geq \|v^*\|_\Omega &\geq \Sigma_{h=1}^{p} |v^*(t_h\chi_I + \tau_h\chi_S) - v^*(t_h\chi_I)| \\
&= \Sigma_{h=1}^{p} \left|\frac{v^*(t_h\chi_I + \tau_h\chi_S) - v^*(t_h\chi_I)}{\tau_h}\right| \lambda(U_h) \\
&= \Sigma_i\Sigma_j \left|\frac{v^*(t_{ij}\chi_I + \tau_{ij}\chi_S) - v^*(t_{ij}\chi_I)}{\tau_{ij}}\right| \lambda(U_{ij}) \\
&= \Sigma_i\Sigma_j \frac{i-1}{k} \lambda(U_{ij}) = \Sigma_i \frac{i-1}{k} \lambda(S_i) \\
&= \Sigma_i \frac{i}{k} \lambda(S_i) - \frac{1}{k} \Sigma_i \lambda(S_i) \geqq \Sigma_i \frac{i}{k} \lambda(S_i) - \frac{1}{k},
\end{aligned}$$

where $S_i = \cup_j U_{ij}$ is as in Lemma 41.6. From Lemma 41.6

we obtain

$$\lambda(S_i + T_i) < \varepsilon;$$

combining this with the previous inequality, we obtain

$$\|v^*\|_\delta \geqq \Sigma_i \frac{i}{k}\lambda(T_i) - \varepsilon\Sigma_i \frac{i}{k} - \frac{1}{k} = \Sigma_i \frac{i}{k}\lambda(T_i) - \varepsilon\frac{k^2(k^2+1)}{2k} - \frac{1}{k}.$$

Letting $\varepsilon \to 0$, we deduce

$$\|v^*\|_\delta \geqq \Sigma_i \frac{i}{k}\lambda(T_i) - \frac{1}{k}.$$

But as $k \to \infty$, we have by Lemma 41.8 that*

$$\Sigma_i \frac{i}{k}\lambda(T_i) \to \int_0^1 |\partial v^*(t)|^+ dt.$$

Since $\frac{1}{k} \to 0$ as $k \to \infty$, we deduce

$$\|v\| \geqq \int_0^1 |\partial v^*(t)|^+ dt.$$

This completes the proof of Lemma 41.11.

<u>LEMMA 41.14</u>. <u>Let</u> $w + r \in BV \cap pNA'$, <u>where</u> $w \in DIAG$. <u>Then for</u> $\delta > 0$ <u>sufficiently small,</u>

*The T_i depend on k as well as on i.

$$\|(w + r)^*\|_\delta \leqq \|r\|.$$

Proof. Since w is in DIAG, it satisfies (19.1). Let k, ζ, and U correspond to w in accordance with (19.1). Let δ be such that

(41.15) $$(\exists t)(\|\zeta(S) - te\| < \delta) \Rightarrow \zeta(S) \in U;$$

this is true for all sufficiently small δ. Let $v = w + r$. For given $\varepsilon > 0$, let Ω be a chain

$$0 = g_0 \leqq g_1 \leqq \cdots \leqq g_m \leqq g_{m=1} = \chi_I$$

of ideal set functions satisfying (41.9), such that

(41.16) $$\|v^*\|_\Omega > \|v^*\|_\delta - \varepsilon.$$

Since $v \in pNA'$, we may find a polynomial in measures $f \circ \nu$, where ν is a vector of measures in NA^+, such that

(41.17) $$\|v - f \circ \nu\|' < \varepsilon/4m;$$

from this and (22.8) it follows that $\|(v - f\circ\nu)^*\|' < \varepsilon/4m$, and hence

(41.18) $$\|(v - f\circ\nu)^*\|_\Omega < 2(m + 1)\varepsilon/4m \leqq \varepsilon.$$

Now apply Lemma 41.4 to the vector measure $\xi = (\zeta, \nu)$, obtaining a chain Γ of sets

$$\emptyset = T_0 \subset T_1 \subset \ldots \subset T_m \subset T_{m+1} = I$$

in $\mathcal{C}$ such that for all i,

$$\xi(T_i) = \int g_i d\xi.$$

It follows that for all i, $\nu(T_i) = \int g_i d\nu$, and so

$$(f\circ\nu)^*(g_i) = f(\int g_i d\nu) = (f\circ\nu)(T_i);$$

hence

$$\|(f\circ\nu)^*\|_\Omega = \|f\circ\nu\|_\Gamma. \tag{41.19}$$

Next, for each i, since g_i satisfies (41.9), there is a number t_i such that

$$\|g_i - t_i\chi_I\| < \delta.$$

Hence for each component ζ_p of ζ,

$$|\zeta_p(T_i)-t_i| = |\int g_i d\zeta_p - t_i| = |\int (g_i - t_i\chi_I)d\zeta_p| \le \int |g_i - t_i\chi_I| d\zeta_p < \delta.$$

Therefore $\|\zeta(T_i) - t_i e\| < \delta$, and so by (41.15), $\zeta(T_i) \in U$; hence $w(T_i) = 0$, and so

(41.20) $$\|w\|_\Gamma = 0.$$

Note also that by (41.17),

(41.21) $$\|v - f\circ\nu\|_\Gamma < 2(m+1)\varepsilon/4m \leqq \varepsilon.$$

Combining (41.16), (41.18), (41.19), (41.20), and (41.21), we get

$$\|v^*\|_\delta \leqq \varepsilon + \|v^*\|_\Omega \leqq \varepsilon + \|(v-f\circ\nu)^*\|_\Omega + \|(f\circ\nu)^*\|_\Omega \leqq 2\varepsilon + \|f\circ\nu\|_\Gamma$$

$$\leqq 2\varepsilon + \|f\circ\nu - v\|_\Gamma + \|v-w\|_\Gamma + \|w\|_\Gamma \leqq 3\varepsilon + \|r\|_\Gamma + 0 \leqq 3\varepsilon + \|r\|.$$

Letting $\varepsilon \to 0$, we obtain the desired result. This completes the proof of Lemma 41.14.

The following proposition is an analogue of Theorem E.

<u>PROPOSITION 41.22</u>. <u>Let</u> $v \in pNAD \cap pNA'$. <u>Then for each</u> $S \in \mathcal{C}$, <u>the derivative</u> $\partial v^*(t, S)$ <u>exists for almost all</u> t <u>in</u> $[0,1]$, <u>and is integrable over</u> $[0,1]$ <u>as a function of</u> t; <u>and</u>

$$(\varphi v)(S) = \int_0^1 \partial v^*(t, S)\,dt.$$

Proof. The proof follows the ideas of the proof of Theorem E (Section 23). Define

$$\Delta_v(t) = \limsup_{\tau\to 0} \frac{v^*(t\chi_I+\tau\chi_S)-v^*(t\chi_I)}{\tau} - \liminf_{\tau\to 0} \frac{v^*(t\chi_I+\tau\chi_S)-v^*(t\chi_I)}{\tau}$$

(cf. (23.5)), and

$$\Delta_v = \int_0^1 \Delta_v(t)dt.$$

From Lemma 41.9, we then obtain

$$0 \leqq \Delta_v \leqq 2\int_0^1 |\partial v^*(t)|^+ dt \leqq 2\|v^*\|_\delta$$

for all $\delta > 0$ (cf. (23.6)); furthermore

$$\Delta_{v+w} \leqq \Delta_v + \Delta_w$$

whenever $v, w \in BV \cap pNA'$. Now let $v \in pNAD \cap pNA'$; for given $\epsilon > 0$, let $v = q + w + r$, where $q \in pNA$, $w \in DIAG$, and $\|r\| < \epsilon$. From Theorem E it follows that $\Delta_q = 0$; hence since $w + r = v - q \in BV \cap pNA'$, we get, using Lemma 41.14, that for δ sufficiently small,

$$0 \leqq \Delta_v \leqq \Delta_q + \Delta_{w+r} \leqq 0 + 2\|(w + r)^*\|_\delta \leqq \|r\| < \epsilon.$$

Letting $\epsilon \to 0$, we deduce $\Delta_v = 0$. Hence $\Delta v(t) = 0$ for almost

all t, i.e., $\partial v^*(t)$ exists a.e. for all v. Whenever it exists we have $|\partial v^*(t)| = |\partial v^*(t)|^+$, and hence by Lemmas 41.11 and 41.5, we have, for all $\delta > 0$,

$$\int_0^1 |\partial v^*(t)|\,dt \leqq \|v^*\|_\delta \leqq \|v^*\| = \|v\|; \tag{41.23}$$

in particular, this implies the integrability of $\partial v^*(t)$.

Now let

$$\theta v = \int_0^1 \partial v^*(t)\,dt;$$

then θv is linear in v, and by (41.23),

$$|\theta v| \leqq \|v^*\|_\delta \leqq \|v\| \tag{41.24}$$

for all $\delta > 0$. For given $\varepsilon > 0$, let $v = q + w + r$, where $q \in pNA$, $w \in DIAG$, and $\|r\| < \varepsilon$. By Theorem E, $\varphi q = \theta q$; therefore by (41.24) and Lemma 41.14,

$$\begin{aligned} |(\varphi v)(S) - \theta v| &\leqq |(\varphi q)(S) - \theta q| + |\varphi(w+r)(S) - \theta(w+r)| \\ &\leqq 0 + |\varphi(w+r)(S)| + |\theta(w+r)| \leqq |\varphi(w+r)(S)| + \|(w+r)^*\|_\delta \\ &\leqq |\varphi(w+r)(S)| + \|r\| < |\varphi(w+r)(S)| + \varepsilon. \end{aligned} \tag{41.25}$$

Now φ coincides with the asymptotic value on pNAD (Proposition (41.1), furthermore, since w and w + r are both in

pNAD, so is r (though it may not be in pNA'). Since $\|\varphi\| = 1$ for the asymptotic value (Proposition 18.1), it follows that

$$\|\varphi r\| \leqq \|r\| < \varepsilon. \tag{41.26}$$

On the other hand, $w \in$ DIAG and φ has the diagonal property (Proposition 19.7); hence $\varphi w = 0$. Combining this with (41.25) and (41.26), we obtain

$$|\varphi v(S) - \theta v| \leqq \|\varphi(w + r)\| + \varepsilon \leqq \|\varphi w\| + \|\varphi r\| + \varepsilon \leqq 2\varepsilon;$$

hence letting $\varepsilon \to 0$, we deduce $\varphi v(S) = \theta v$, as was to be proved. This completes the proof of Proposition 41.22.

The following is an analogue of Proposition 27.8.

PROPOSITION 41.27. *If** $w \in$ pNA', *then every member of the core of* w *is in* NA.

Proof. The proof follows the ideas of that of Proposition 27.8. What is needed is the existence of an NA^+ measure ν such that if $T_1, T_2, \ldots$ is a sequence of sets in $\mathcal{C}$ with $\nu(T_i) \to 0$, then (27.10) holds, i.e.,

$$w(T_i) \to 0 \quad \text{and} \quad w(I \backslash T_i) \to w(I);$$

*The same conclusion holds when $w \in$ pNAD.

this may be called* the <u>continuity of</u> w <u>w.r.t.</u> ν <u>at</u> $\emptyset$ <u>and</u> I. To establish this, let $w_1, w_2, \ldots$ be a sequence in pNA such that

$$\|w - w_j\|' \to 0.$$

Let ν_j be probability measures in NA such that $w_j \ll \nu_j$ for all j. Set $\nu = \Sigma_{i=1}^{\infty} \nu_j/2^j$. If $\nu(T_i) \to 0$, then $\nu_j(T_i) \to 0$ for all j, and hence as $i \to \infty$,

$$w_j(T_i) \to 0 \quad \text{and} \quad w_j(I\backslash T_i) \to w_j(I).$$

For given $\epsilon > 0$, pick j such that $\|w - w_j\|' < \epsilon/2$, and let N be such that whenever $i > N$,

$$|w_j(T_i)| < \epsilon/2 \text{ and } |w_j(I\backslash T_i) - w_j(I)| < \epsilon/2.$$

Then whenever $i > N$,

$$|w(T_i)| < \epsilon \quad \text{and} \quad |w(I\backslash T_i) - w(I)| < \epsilon,$$

*It is clear how this definition may be generalized to cover continuity of w w.r.t. ν at an arbitrary S. Compare the discussion of Example 33.11, where continuity of a set function at S is defined without referring to ν. It may be seen that continuity at S is implied by continuity at S w.r.t. some ν in NA^+; and this, in turn, is implied by absolute continuity.

i.e., (27.10) holds. Thus the desired continuity property is established. The remainder of the proof is exactly as in Proposition 27.8.

The following proposition is an analogue of Theorem F.

PROPOSITION 41.28. Let v be a superadditive set function in pNAD ∩ pNA' that is homogeneous of degree 1. Then there is a unique point in the core of v, namely φv.

Proof. The proof follows the lines of the proof of Theorem F rather closely. Results analogous to Lemmas 27.2, 27.4, 27.5, and Corollary 27.3 are readily established, the only difference in the proofs being that references to Theorem E must be replaced by references to Proposition 41.22. Thus it is established that φv is in the core of v, and that it is the only member of the core of v that is in NA. But by Proposition 41.27, there are no other members of the core of NA, and so the proof of Proposition 41.28 is complete.

With this, the first stage in the proof of Proposition 41.2 is complete. To finish the proof of Proposition 41.2, we must show that the v of that proposition is well-defined, that it is in pNAD ∩ pNA', and that it is superadditive and homogeneous of degree 1.

Let $\mathcal{D}$ be any subset of the σ-field $\mathcal{C}$ of coalitions, such that ∅ and I are in $\mathcal{D}$. It is not required that $\mathcal{D}$

be a σ-field. A real-valued function w on $\mathcal{D}$ with $w(\emptyset) = 0$ is called a $\mathcal{D}$-function. Monotonicity, bounded variation, and the variation norm are defined for $\mathcal{D}$-functions just as they are for set-functions. Thus a $\mathcal{D}$-function w is monotonic if $S_1 \supset S_2 \in \mathcal{D}$ imply $w(S_1) \geq w(S_2)$; w is of bounded variation if it is the difference of monotonic functions; and in that case, its (variation) norm $\|w\|$ is defined by

$$\|w\| = \inf (w_1(I) + w_2(I)), \tag{41.29}$$

where the inf is taken over all monotonic $\mathcal{D}$-functions w_1 and w_2 such that $w = w_1 - w_2$.

LEMMA 41.30. *If w is a $\mathcal{D}$-function of bounded variation, then*

$$\|w\| = \sup \Sigma_{i=1}^{k} |w(S_i) - w(S_{i-1})|,$$

where the sup is taken over all chains

$$\emptyset = S_0 \subset \dots \subset S_k = I$$

of S_i in $\mathcal{D}$. Furthermore, the inf in the definition (41.29) *of $\|w\|$ is attained.*

Proof. The proof is similar to that of Proposition 4.1.

LEMMA 41.31. *Let* w *be a* $\mathcal{B}$*-function of bounded variation. Then there exists a set function* v *in* BV *such that* $v|\mathcal{B} = w$ *and*

$$\|v\| = \|w\|.$$

Proof. First let w be monotonic. Define v by

$$v(S) = \sup\{w(T) : T \in \mathcal{B} \quad \text{and} \quad T \subset S\}.$$

Clearly v is monotonic, and $v(I) = w(I)$. This completes the proof in case w is monotonic.

In the general case, using Lemma 41.30, let w_1 and w_2 be monotonic $\mathcal{B}$-functions such that

$$w = w_1 - w_2$$

and $w_1(I) + w_2(I) = \|w\|$. Let v_1 and v_2 be monotonic set functions such that $v_1|\mathcal{B} = w_1$, $v_2|\mathcal{B} = w_2$, and $v_1(I) = w_1(I)$, $v_2(I) = w_2(I)$. Define $v = v_1 - v_2$. Then

$$\|v\| \leq \|v_1\| + \|v_2\| = w_1(I) + w_2(I) = \|w\|.$$

But again from Lemma 41.30 it is clear that $\|v\| \geq \|w\|$, since v is an extension of w. This completes the proof of Lemma 41.31.

Throughout the remainder of this section, $\underset{\sim}{a}$ will be a fixed μ-integrable function from I to Ω, u will be a fixed member of $\mathcal{U}_1$, and v will be the set-function corresponding to $\underset{\sim}{a}$ and u (i.e., $v(S) = u_S(\int_S \underset{\sim}{a})$); that it is well-defined follows from Proposition 36.1. We will use the notation $\hat{u}$ for δ-approximations to u (though $\hat{u}$ is not fixed throughout the discussion), and for given $\hat{u}$, we will denote by $\hat{v}$ the set function corresponding to $\underset{\sim}{a}$ and $\hat{u}$ (i.e., $\hat{v}(S) = \hat{u}_S(\int_S \underset{\sim}{a})$). Finally, φ will continue to denote the unique value on pNAD given by Proposition 41.1.

<u>LEMMA 41.32</u>. <u>For each</u> $\delta > 0$, <u>define</u>

$$\mathcal{D}_\delta = \{S \in \mathcal{C} : \|\int_S \underset{\sim}{a} - \mu(S)\int \underset{\sim}{a}\| < \delta\}.$$

<u>Then for every</u> $\varepsilon > 0$ <u>there is a</u> $\delta > 0$ <u>such that if</u> $\hat{u} \in \mathcal{U}_1$ <u>is a</u> δ-<u>approximation to</u> u, <u>then</u>

$$\|v|\mathcal{D}_\delta - \hat{v}|\mathcal{D}_\delta\| < \varepsilon .$$

<u>Proof</u>. The proof is similar to that of Proposition 40.24, the restriction to $\mathcal{D}_\delta$ taking the place of condition (40.25). W.l.o.g.* let $\int \underset{\sim}{a} = e$. Let δ_1 correspond to $\frac{1}{4}\varepsilon$ in accordance with Lemma 40.22; w.l.o.g. let $\delta_1 < 1$. Let

*Any commodity j for which $\int \underset{\sim}{a}^j = 0$ may simply be excluded from consideration.

$$\epsilon_2 = \min\left(\frac{1}{2}\delta_1, \frac{1}{2}\epsilon\right),$$

and choose δ_2 to correspond to ϵ_2 in accordance with Lemma 40.1. Let $\delta = \min\left(\frac{1}{8}\delta_1, \delta_2\right)$.

Let $w = v - \hat{v}$, and let

$$\emptyset = S_0 \subset S_1 \subset \ldots \subset S_m \subset S_{m+1} = I$$

be a chain in $\mathcal{D}_\delta$. By Lyapunov's theorem in n dimensions, it is always possible to insert finitely many additional sets

$$S_{01}, S_{02}, \ldots, S_{11}, S_{12}, \ldots, \ldots, S_{m1}, S_{m2}, \ldots$$

in $\mathcal{D}_\delta$ into the chain so that

$$S_0 \subset S_{01} \subset S_{11} \subset \ldots \subset S_1 \subset S_{11} \subset S_{12} \subset \ldots \subset \ldots \subset S_m \subset S_{m1}$$
$$\subset S_{m2} \subset \ldots \subset S_{m+1}$$

and the measure of the difference between two neighboring sets is $< \delta$; that is, if we relabel the new sequence $U_0, \ldots, U_{p+1} = I$, then $\mu(U_{k+1} \setminus U_k) < \delta$ for all k. Furthermore, by Lyapunov's theorem (in n dimensions) we may suppose w.l.o.g. that for one of the U_k, say for U_q, we have*

*The one-dimensional Lyapunov theorem is not sufficient because we must make sure that $U_q \in \mathcal{D}_\delta$.

$\int_{U_q} \underset{\sim}{a}^1 = \frac{3}{4}\delta_1$. From $U_q \in \mathscr{D}_\delta$ it follows that for all j,

$$\left|\int_{U_q} \underset{\sim}{a}^j - \int_{U_q} \underset{\sim}{a}^1\right| < 2\delta \leq \frac{1}{4}\delta_1,$$

and hence

$$\int_{U_q} \underset{\sim}{a}^j < \frac{3}{4}\delta_1 + \frac{1}{4}\delta_1 = \delta_1 \tag{41.33}$$

and

$$\int_{U_q} \underset{\sim}{a}^j > \frac{3}{4}\delta_1 - \frac{1}{4}\delta_1 = \frac{1}{2}\delta_1 \geqq \epsilon_2. \tag{41.34}$$

Now

$$\Sigma_{k=0}^m |w(S_{k+1}) - w(S_k)| \leqq \Sigma_{k=0}^p |w(U_{k+1}) - w(U_k)| = \Sigma_{k=0}^{q-1} + \Sigma_{k=q}^p.$$

From (41.34), Lemma 40.1, and the fact that $\hat{u}$ is a δ-approximation--hence a fortiori a δ_2-approximation--to u, we then obtain

$$\Sigma_{k=q}^p \leqq \epsilon_2 \leqq \frac{1}{2}\epsilon.$$

Furthermore, since u is a δ-approximation to u, it is a fortiori a δ_1-approximation. Hence by the monotonicity of v and $\hat{v}$ and by (41.33) and Lemma 40.24, we have, exactly as in the proof of Proposition 40.24, that

$$\Sigma_{k=0}^{q-1} \leq v(U_q) + \hat{v}(U_q) < \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \frac{1}{2}\varepsilon .$$

We conclude that

$$\Sigma_{k=0}^{m} |w(S_{k+1}) - w(S_k)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon ,$$

and it follows that

$$\|v|\mathcal{D}_\delta - \hat{v}|\mathcal{D}_\delta\| = \|w|\mathcal{D}_\delta\| < \varepsilon .$$

This completes the proof of Lemma 41.32.

<u>COROLLARY 41.35</u>. $v \in$ pNAD.

<u>Proof</u>. Let ε be given, and let δ and $\hat{u}$ correspond to ε in accordance with Lemma 41.32. By Lemma 41.31, there then exists a set-function r in BV such that $r|\mathcal{D}_\delta = (v - \hat{v})|\mathcal{D}_\delta$ and

$$\|r\| = \|(v - \hat{v})|\mathcal{D}_\delta\| < \varepsilon .$$

Then

$$v - r = (v - \hat{v} - r) + \hat{v}.$$

Now $(v - \hat{v} - r)|\mathcal{D}_\delta = 0$, and hence $v - \hat{v} - r \in$ DIAG; and by Lemma 39.16, $\hat{v} \in$ pNA. Thus

$$v - r \in pNA + DIAG.$$

Since $\|r\| < \varepsilon$ and ε was chosen arbitrarily small, it follows that v is in the closure of pNA + DIAG. This completes the proof of Corollary 41.35.

Let H' be the set of all superadditive set functions in pNA' that are homogeneous of degree 1.

<u>PROPOSITION 41.36</u>. H' <u>is closed in the supremum norm</u>.

<u>Proof</u>. The proof is exactly analogous to that of Proposition 27.12.

<u>PROPOSITION 41.37</u>. $v \in H'$.

<u>Proof</u>. Let $\varepsilon > 0$ be given, let δ correspond to $\varepsilon/(1 + \Sigma \int \underset{\sim}{a})$ in accordance with Proposition 37.11, and let $\hat{u} \in \mathcal{U}_1$ be a δ-approximation to u. Then for all $S \in \mathcal{C}$ we have

$$|v(S) - \hat{v}(S)| = |u_S(\textstyle\int_S \underset{\sim}{a}) - \hat{u}_S(\textstyle\int_S \underset{\sim}{a})| < \varepsilon \frac{1 + \Sigma\int_S \underset{\sim}{a}}{1 + \Sigma\int \underset{\sim}{a}} \leq \varepsilon .$$

Hence

$$\|v - \hat{v}\|' = \sup_S |v(S) - \hat{v}(S)| < \varepsilon .$$

Since an appropriate $\hat{u}$ can be found (Proposition 35.6)

and since $\hat{v} \in H$ (Lemma 39.16), it follows that v is in the closure of H in the supremum norm. But this is certainly included in the sup-closure of H', and so by Proposition 41.36 in H'. The proof of Proposition 41.37 is complete.

Proposition 41.2 follows immediately from Propositions 41.28, 41.35, and 41.37; Proposition 31.7 follows immediately from Propositions 41.1 and 41.2.

42. POSSIBILITIES FOR EXTENSIONS OF THE MAIN RESULTS

A. The Mixing Value

In Section 31 the question was raised as to whether Proposition 31.7 can be proved when the mixing value is substituted for the asymptotic value. Here we will show that this is indeed the case if it can be shown that v (as defined in (30.1)) is in AC. Of course, if $v \notin AC$, then $v \notin MIX$, so that there is then no hope for proving the analogue of Proposition 31.7. We do not know whether or not $v \in AC$.

PROPOSITION 42.1. $pNAD \cap AC \subset MIX$.

Proof. The proof is patterned after that of Proposition 19.3 (which states, among other things, that $DIAG \cap AC \subset MIX$). Let $v \in pNAD \cap AC$. Let $\{\epsilon_1, \epsilon_2, \ldots\}$ be a sequence tending to 0; then for each j we may find a decomposition of v (depending on j),

$$v = v_1 + v_2 + v_3,$$

where $v_1 \in pNA$, $v_2 \in DIAG$, and $\|v_3\| \leq \epsilon_j$. Since $v_1 \in pNA \subset MIX$ (Proposition 16.9), there is a probability measure μ_{v_1} corresponding to v_1 in accordance with Proposition 14.1. For each j, let k, ζ, and U (which, of course, depend on j) correspond to v_2 in accordance with (19.1), and let

$$\mu^j = \mu_{v_1} + \frac{1}{k}\Sigma_{i=1}^k \zeta_i.$$

Let

$$\mu_v = \Sigma_{j=1}^\infty \frac{1}{2^j}\mu^j.$$

Let μ be a probability measure in NA such that $\mu_v << \mu$, let $\{\Theta_1, \Theta_2, \ldots\}$ be a μ-mixing sequence, and let $\mathcal{R}$ be a measurable order. Now fix j for the time being; note that $\mu_{v_1} << \mu$ and that $\zeta_i << \mu$ for all i. Proceeding exactly as in the proof of Proposition 19.3, we conclude that there is a positive integer m_0 such that

$$\zeta(I(s; \Theta_m\mathcal{R})) \in U$$

for all $s \in I$ and $m > m_0$. It follows that for all such s and m, we have

$$v_2(I(s; \Theta_m\mathcal{R})) = 0. \tag{42.2}$$

For fixed $m > m_0$, let $\mathcal{Q} = \Theta_m\mathcal{R}$, and let W be in the field (not σ-field) $H(\mathcal{Q})$ generated by the initial segments $I(s; \mathcal{Q})$ (cf. the proof of Proposition 12.7). Then W can be written in the form

$$W = \cup_{i=1}^p [t_i, s_i)_{\mathcal{Q}},$$

i.e., as a finite union of disjoint $\mathcal{Q}$-intervals, where

(42.3) $$s_p \mathcal{Q} t_p \mathcal{Q} \ldots \mathcal{Q} s_1 \mathcal{Q} t_1.$$

Since $v_2 + v_3 = v - v_1 \in AC$, it follows that $\varphi(v_2 + v_3; \mathcal{Q})$ exists (Proposition 12.7), and so by (12.2) and 42.2),

$$|\varphi(v_2 + v_3; \mathcal{Q})(W)| = |\varphi(v_2 + v_3; \mathcal{Q})(\cup_{i=1}^{p}[t_i, s_i)_{\mathcal{Q}})|$$

$$= |\Sigma_{i=1}^{p} [\varphi(v_2 + v_3; \mathcal{Q})(I(s_i; \mathcal{Q})) - \varphi(v_2 + v_3; \mathcal{Q})(I(t_i; \mathcal{Q}))]|$$

$$= |\Sigma_{i=1}^{p} [(v_2 + v_3)(I(s_i; \mathcal{Q})) - (v_2 + v_3)(I(t_i; \mathcal{Q}))]|$$

$$= |\Sigma_{i=1}^{p} [v_3(I(s_i; \mathcal{Q})) - v_3(I(t_i; \mathcal{Q}))]|.$$

By (42.3), the last expression is the variation of v_3 over a subchain of a certain chain, and so it is $\leqq \|v_3\| \leqq \epsilon_j$. Thus we have

$$|\varphi(v_2 + v_3; \Theta_m\mathcal{R})(W)| \leqq \epsilon_j$$

whenever $W \in H(\Theta_m\mathcal{R})$; but since the field $H(\Theta_m\mathcal{R})$ generates the σ-field $\mathcal{C}$ (by (12.3)), it follows from a standard approximation argument* that

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*One uses $[H_1]$, p. 56, Theorem D. Compare the end of the proof of Theorem 12.7.

$$|\varphi(v_2 + v_3;\ \Theta_m\mathcal{P})(S)| \leq \varepsilon_j$$

for all $S \in \mathcal{C}$; of course this holds for all $m > m_0$. From this, $v_1 \in$ MIX (Proposition 16.9), and Proposition 14.1, it follows that for all $S \in \mathcal{C}$,

$$\begin{aligned}
&\limsup_{m\to\infty} \varphi(v;\ \Theta_m\mathcal{P})(S) \\
&\leq \lim \varphi(v_1;\ \Theta_m\mathcal{P})(S) + \limsup \varphi(v_2 + v_3;\ \Theta_m\mathcal{P})(S) \\
&\leq (\varphi v_1)(S) + \varepsilon_j,
\end{aligned}$$

where v_1 is the mixing value (or, for that matter, the unique value on pNA). Similarly

$$\liminf_{m\to\infty} \varphi(v;\ \Theta_m\mathcal{P})(S) \geq (\varphi v_1)(S) - \varepsilon_j.$$

Hence

$$\limsup \varphi(v;\ \Theta_m\mathcal{P})(S) - \liminf \varphi(v;\ \Theta_m\mathcal{P})(S) \leq 2\varepsilon_j.$$

Now the left side of this inequality is independent of j; so we may let $j \to \infty$, and conclude that $\lim_{m\to\infty} \varphi(v;\ \Theta_m\mathcal{P})(S)$ exists. By Proposition 14.1, this completes the proof of Lemma 42.1.

PROPOSITION 42.4. <u>There is exactly one value on</u> pNAD $\cap$ AC <u>that is continuous in the variation norm and enjoys the diagonal property. This value coincides with the mixing value on</u> pNAD $\cap$ AC, <u>as well as with the value</u> φ <u>of Proposition</u> 41.1

<u>Proof</u>. Since we have shown that pNAD $\cap$ AC $\subset$ MIX, it is only necessary to establish the uniqueness in the first sentence of the proposition. Now

$$\begin{aligned} \text{pNAD} \cap \text{AC} &= \overline{(\text{pNA} + \text{DIAG})} \cap \overline{\text{AC}} = \overline{(\text{pNA} + \text{DIAG}) \cap \text{AC}} \\ &= \overline{\text{pNA} + (\text{DIAG} \cap \text{AC})}, \end{aligned}$$

since pNA $\subset$ AC. The proof of uniqueness is now completed just as in Proposition 41.1, except that DIAG $\cap$ AC must be substituted for DIAG. This completes the proof of Proposition 42.4.

Suppose now that the hypotheses of Proposition 31.7 are satisfied, and suppose further that $v \in$ AC. Then by Proposition 41.2, v is well-defined, $v \in$ pNAD $\cap$ AC, and the core of v consists of the single point φv, where φ is the value of Proposition 41.1. But by Proposition 42.4, φv coincides with the mixing value of v as well. Thus we have shown that if $v \in$ AC, then the mixing value of v exists and is the unique point in the core of v.

B. Positivity of Initial Resources

Propositions 31.5, 31.7, 33.2, and 40.26 give conditions under which the positivity condition (condition (31.4)) can be dispensed with. It is, however, possible that it can be dispensed with altogether; i.e., that Theorem G remains true if this condition is simply dropped, without substituting anything for it. We have not been able either to prove or disprove this possibility.

C. Strict Monotonicty of u

Our proofs make extensive use, especially in Section 37, of the assumption that each of the u(x, s) be strictly increasing in x. It is, however, possible that a careful treatment might be able to dispense with this assumption, particularly under certain conditions (such as (31.4)). Compare Proposition 2.2 of [A-P], which would also be considerably easier to prove if one would assume that the u-functions are strictly increasing in x.

D. Attainment of the Max in the Definition of v

This question was treated via a number of examples in Subsection D of Section 33, where we showed that it is hopeless to try to extend our results to the case in which the max is not attained, at least in v(I). The question arises, though, whether it is necessary to assume the asymptotic condition (31.2), or whether it would not be enough simply to assume that

(42.5) $v(I)$ is attained

or

(42.6) all the $v(S)$ are attained

or something of that nature.

Aesthetically, an "explicit" condition like (31.2) is preferable to a condition like (42.5), since given a specific family of u-functions, it may be difficult to tell whether (42.5) holds. The mathematical question of whether (31.2) can be replaced by (42.5) still remains, though.

Our method of proof is based on approximations by u's of finite type, this is based on the norm on $\mathfrak{Z}_1$ defined in Section 35, and this in turn depends essentially on (31.2). Thus (31.2) is used not only to establish that the $v(S)$ are attained, but also directly in the proof. It appears that if one wishes to substitute (42.5) or (42.6), one would need an entirely new line of proof. We do not, of course, have a counterexample.

The most natural candidate for a replacement for (31.2) would seem to be neither (42.5) nor (42.6), but rather the stronger

(42.7) $u_S(a)$ is attained for all $S \in \mathcal{C}$ and all $a \in \Omega$.

Condition (42.7) is equivalent to the condition that for any $S \in \mathcal{C}$, the integral of the subgraphs* of $u(\cdot, s)$ over $s \in S$ be closed. Our problem is unsolved when any one of (42.5), (42.6), or (42.7) is substituted for (31.2).

In the extremely special case in which u is of finite type and all the $u(\cdot, s)$ are concave, the methods of Section 39 can probably be pushed through; that is to say, though we have not checked the details, we believe that in this case Theorem G can be proved without assumptions (31.2) and (31.4). In any event, (42.7) holds in this case, because the subgraph of the function u_S is a finite sum of closed subgraphs; this must be closed, since all the subgraphs are in the nonnegative orthant.**

E. Dispensing with Assumption (2.1)

Assumption (2.1), according to which $(I, \mathcal{C})$ is isomorphic to the unit interval with the Borel σ-field, is needed in this part only because without it there may be more than one value on pNA. Thus Proposition 31.7 remains true as it stands even without (2.1), and Theorem G and Proposition 31.5 remain true if "the value φv" appearing

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*The subgraph of a nonnegative function f is here defined to be $\{(x, y) : 0 \leq y \leq f(x)\}$. In Section 37 we used the same term for the set $\{(x, y) : y \leq f(x)\}$; usually it doesn't make much difference which way one defines this term, but here the condition $y \geq 0$ is convenient, as we shall see below.

**This is the reason for defining the subgraph by $0 \leq y \leq f(x)$ rather than just by $y \leq f(x)$.

at the end of the statement of Theorem G is interpreted to be that* value for which $\varphi(\mu^k) = \mu$ for all NA probability measures μ and all k.

The theorems of [A-P] that we have quoted here depend on the theory of integrals of set-valued functions $[A_7]$, this in turn depends on a selection theorem of von Neumann [VN, p. 448, Lemma 5], and this in turn depends on (2.1). But von Neumann's theorem can be generalized $[A_6]$ so as not to depend on (2.1), so for the purpose of applying the results of [A-P], (2.1) is not needed.

*It can be seen by the methods of Part I that there is a unique such value, and that it obeys the integral formula (3.1).

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